Does the New Keynesian Model Have a Uniqueness Problem?*

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Abstract

This paper addresses whether non-uniqueness of equilibrium is a substantive problem for policy analysis in New-Keynesian (NK) models. There would be a substantive problem if there were no compelling way to select among different equilibria that give different answers to critical policy questions. In fact there is: stability-under-learning. We focus our analysis on the efficacy of fiscal policy when the economy is in the ZLB. We study a fully non-linear NK model with Calvo-pricing frictions and argue that the model has a unique stable-under-learning rational expectations equilibrium. In that equilibrium, the implications of the model for fiscal policy inherit all of the key properties of linearized NK models.

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1. Introduction

New Keynesian (NK) models have been enormously influential in terms of their policy implications[1]. The models' implications for fiscal policy are particularly striking when the zero lower bound (ZLB) on the nominal rate of interest is binding.[2] Eggertsson and Woodford (2003) (EW) and Eggertsson (2004) develop an elegant and transparent framework for studying fiscal policy in the NK model at the ZLB.

The key results that emerge from the literature can be summarized as follows[3]. First, when the ZLB binds, the fall in output is potentially very large. Second, the output multiplier associated with government consumption is larger when the ZLB binds than when it does not bind. Third, the more flexible are prices and the longer is the expected duration of the ZLB, the larger is the drop in output and the larger is the government consumption multiplier.

These controversial results are based on linearized versions of the NK model, which has a unique solution. In fact, the non-linear NK models have multiple equilibria, even if one restricts attention, as did EW, to minimum state variable ZLB equilibria. As stressed by Mertens and Ravn (2015), policy prescriptions can vary a great deal across those equilibria. At some ZLB equilibria, the government consumption multiplier (or 'multiplier' for short) is small or even negative. In others, it is very large. So, in principle, non-uniqueness of equilibria poses an enormous challenge for policy analysis based on NK models.

This paper addresses a simple question: is non-uniqueness of equilibria a *substantive* problem for policy analysis in NK models? There would be a substantive

[1]For a classic exposition of the NK model see Woodford (2003.)

[2]It is widely understood that zero is not the critical lower bound. What is critical is that some lower bound on the interest rate becomes binding on monetary policy.

[3]See, for example, EW, Eggertsson (2011) and Christiano, Eichenbaum and Rebelo (2011) (CER),

problem if there were no compelling way to select among different equilibria that give different answers to critical policy questions. To be concrete we focus our analysis on the impact of changes in government consumption when the economy is at the ZLB.

Our argument starts from the presumption that the assumption of rational expectations is *obviously* wrong. But it can be a useful modeling strategy for thinking about a world where the strong assumptions associated with rational expectations aren't literally satisfied.[4] In the spirit of the literature summarized by Evans and Honkapohja (2001), we adopt the following selection criterion for rational expectations equilibria. Suppose agents make a 'small' error in forming expectations about variables relative to their values in a particular rational expectations equilibrium. Would the economy converge to a rational expectations equilibrium if agents form expectations using simple learning rules? If yes, then we say the rational expectations equilibrium is stable-under-learning, or for short, learnable. From this perspective, stability-under-learning is a necessary condition for an equilibrium and the associated policy implications to be empirically interesting. Rational expectations equilibria that aren't learnable are best treated as mathematical curiosities.[5]

[4] Indeed that is how Lucas viewed it: "... the model described above 'assumes' that agents know a great deal about the structure of the economy and perform some non-routine computations. It is in order to ask, then: will an economy with agents armed with 'sensible' rules-of-thumb, revising these rules from time to time so as to claim observed rents, tend as time passes to behave as described..." Lucas (1978).

[5] Our notion of stability contrasts sharply with the one used in Benhabib, Schmidt Grohe and Uribe (BSGU) (2001). They study an endowment economy populated by a representative household with preferences defined over consumption and real balances. There is also a monetary authority that sets the nominal interest rate using a Taylor rule subject to a ZLB constraint. BSGU show that there are two steady states corresponding to a nonnegative inflation rate and a deflation rate equal to the household's gross discount rate. They consider the scenario in which actual inflation π_t is close to but not exactly equal to either of the steady state values. BSGU say that a steady state is stable if the economy converges back to it in a rational expectations equilibrium. BSGU show that the deflation steady state equilibrium is the unique stable steady state in their sense of the term.

We apply the stable-under-learning criterion to a standard fully non-linear NK model with Calvo-pricing frictions. Working with this model poses two interesting challenges. First, unlike linearized NK models of the type considered by EW, the rational expectations equilibria can't be characterized by a set of numbers. Because there is an endogenous state variable (price dispersion), finding an equilibrium amounts to finding a set of functions. Second, we must think about how agents learn about these functions.

Our basic results can be summarized as follows. First, we find that there are multiple rational expectations equilibria, including sunspot equilibria. When we consider fundamental shocks that trigger ZLB episodes, we find two minimum state variable rational expectations equilibria at the ZLB. These equilibria converge to different deflation rates if the ZLB episode lasts forever. Second, consistent with Mertens and Ravn (2015), the impact of government consumption can be very different in the different equilibria when the ZLB binds. For example, there exist both sunspot and minimum state variable rational expectations equilibria in which the multiplier is actually negative. Third, we argue that there exists a unique interior equilibrium in the non-linear Calvo model that is stable-under-learning. Fourth, and most importantly, the controversial predictions of the linearized NK model about fiscal policy at the ZLB, including the large size of the multiplier at the ZLB, are satisfied at the unique learnable rational expectations equilibrium. That equilibrium is the one that converges to a relatively low ZLB deflation rate. Based on this analysis we conclude that the Calvo model does not have a substantive uniqueness problem, as least for the analysis of fiscal policy at the ZLB.

Many authors have used non-linear versions of the Rotemberg (1982) model of nominal price rigidities to proxy for the Calvo model. In the Rotemberg model the representative firm faces a quadratic cost of adjusting nominal prices. Linear approximations to the Calvo and the Rotemberg models give rise to the same set of equations whose solution defines an equilibrium. In contrast, non-linear versions of the model are potentially very different. As it turns out, some of the non-linear properties of the Rotemberg model are very sensitive to the details of how one formulates adjustment costs for prices. Specifically, we show that the number of equilibria at the ZLB and their stability properties depend on whether and exactly how one scales adjustment costs for growth. Remarkably, we still always find that there is only one equilibrium at the ZLB that is stable-under-learning. Moreover, all of the predictions of the log-linear NK for the impact of fiscal policy in the ZLB hold at that equilibrium. Indeed, for our benchmark parametrizations, the value of the multiplier in the linear and non-linear model are extremely similar.

As a by-product of our analysis, we use our non-linear model to assess the robustness of policy implications about fiscal policy at the ZLB that have been derived using log linear approximations to the NK model. We find that linear approximations work quite well for assessing the size of the multiplier and the drop in GDP that occurs in the ZLB. Evidence that the quality of linear approximations is poor rests on examples where output deviates by more than roughly 20 percent from its steady state, cases where no one would expect linear approximations to work well. There is one interesting difference between the linear and non-linear models. It is well know that for some parameter values, the multiplier in the linear model shoots off to infinity, say as the expected length of the ZLB episode becomes large or prices become very flexible (see for example CER (2011)). For the same parameter values, these extreme results manifest themselves in a different way in the non-linear Calvo model: equilibrium simply ceases to exist.

The Great Recession was a very unusual event. The learning equilibria underlying our stability calculations are of interest as a way of modeling how agents behave in the wake of an unusual shock that pushes the economy into a prolonged ZLB episode. So we analyze the impact of an increase in government consumption along the learning equilibrium. Our findings here can be summarized as follows. First, the learning equilibrium is unique. Second, the size of the multiplier is large in the learning equilibrium. The latter finding is different than results reported in Mertens and Ravn (2015). As it turns out the main reason for the difference in our results is that despite their backwards looking learning rule, Mertens and Ravn change agents expectations about future consumption and inflation when they change government consumption. We do not.

The remainder of this paper is organized as follows. In section two we analyze rational expectations equilibra at the ZLB in a nonlinear Calvo model. We also assess the quality of linear approximations to the Calvo model. Section three contains our main results regarding stability-under-learning of the equilibria we find at the ZLB. In section four we discuss learning equilibria. Section five contains our analysis of the non-linear Rotemberg model. Concluding remarks are contained in section six.

2. Fiscal Policy in the ZLB

In this section we derive the implications of the NK model for the effects of changes in government purchases when the ZLB in binding. We conduct our analysis in a non-linear version of the NK model in which firms face Calvo price-setting frictions.

2.1. Model Economy

A representative household maximizes

$$E_0 \sum_{t=0}^{\infty} d_t \left[\log \left(C_t \right) - \frac{\chi}{2} h_t^2 \right]$$

where C_t is consumption, h_t are hours worked, and

$$d_0 = 1,\ d_t = \prod_{j=1}^{t} \left(\frac{1}{1 + r_{j-1}} \right),\ t \geq 1.$$

Here, d_t is the household's rate of time discounting. As in EW, the rate of time discounting varies over time as r_t changes. EW refer to r_t as the natural rate of interest. We assume that r_t can take on two values: r and r^ℓ, where $r^\ell < 0$.[6] The stochastic process for r_t is given by

$$\Pr\left[r_{t+1} = r^\ell | r_t = r^\ell\right] = p,\ \Pr\left[r_{t+1} = r | r_t = r^\ell\right] = 1 - p,\ \Pr\left[r_{t+1} = r^\ell | r_t = r\right] = 0. \tag{2.1}$$

We assume that r_t is known at time t. The household faces the budget constraint

$$P_t C_t + B_t \leq \left(1 + R_{t-1}\right) B_{t-1} + W_t h_t + \Pi_t.$$

Here P_t is the price of the consumption good, B_t are the quantity of risk-free nominal bonds, R_{t-1} is the gross nominal interest rate paid on bonds held from period $t-1$ to period t, W_t is the nominal wage, and Π_t represents lump-sum profits net of lump-sum government taxes.

The two first order necessary conditions associated with an interior solution to the household's problem are:

$$\chi h_t C_t = \frac{W_t}{P_t}, \tag{2.2}$$

and

$$\frac{1}{1 + R_t} = \frac{1}{1 + r_t} E_t \frac{P_t C_t}{P_{t+1} C_{t+1}}. \tag{2.3}$$

[6] We use the letter ℓ to indicate that the natural rate of interest is 'low'.

A final homogeneous good, Y_t, is produced by competitive and identical firms using the technology:

$$Y_t = \left[\int_0^1 (Y_{j,t})^{\frac{\varepsilon}{\varepsilon-1}} \, dj \right]^{\frac{\varepsilon-1}{\varepsilon}}, \tag{2.4}$$

where $\varepsilon > 1$. The representative firm chooses inputs, $Y_{j,t}$, to maximize profits:

$$P_t Y_t - \int_0^1 P_{j,t} Y_{j,t} dj,$$

subject to the production function (2.4). The firm's first order condition for the j^{th} input is:

$$Y_{j,t} = (P_t / P_{j,t})^{\varepsilon} \, Y_t. \tag{2.5}$$

The j^{th} input in (2.4) is produced by firm j who is a monopolist in the product market and is competitive in factor markets. Monopolist j has the production function:

$$Y_{j,t} = h_{j,t}. \tag{2.6}$$

Here, $h_{j,t}$ is the quantity of labor used by the j^{th} monopolist. The monopolist maximizes

$$E_0 \sum_{t=0}^{\infty} d_t \lambda_t \left((1 + \upsilon) P_{j,t} - P_t s_t \right) Y_{j,t}. \tag{2.7}$$

The j^{th} monopolist sets its price, $P_{j,t}$, subject to the demand curve, (2.5), and the following Calvo sticky price friction (2.8):

$$P_{j,t} = \begin{cases} P_{j,t-1} & \text{with probability } \theta \\ \tilde{P}_{j,t} & \text{with probability } 1 - \theta \end{cases}. \tag{2.8}$$

Here $\tilde{P}_{j,t}$ is the price chosen by the monopolist j in the event that he can re-optimize his price. The variable υ is a subsidy designed to remove monopoly power distortions in a deterministic steady state when $r_t = r$. The monopolist satisfies whatever demand occurs at its posted price. The real marginal cost facing each monopolist is given by:

$$s_t \equiv \frac{W_t}{P_t} = \chi h_t C_t. \tag{2.9}$$

Since all monopolists face the same problem, $\tilde{P}_{j,t}$ is independent of j and we denote its value by $\tilde{P}_t$. The first-order condition of monopolist j can be written as

$$\tilde{p}_t = \frac{\tilde{P}_t}{P_{t-1}} = \pi_t \frac{K_t}{F_t}$$

where K_t and F_t are infinite sums that can be written recursively as

$$K_t = \frac{Y_t}{C_t} s_t + \theta \frac{1}{1+r_t} E_t \pi_{t+1}^{\varepsilon} K_{t+1}$$

and

$$F_t = \frac{Y_t}{C_t} + \theta \frac{1}{1+r_t} E_t \pi_{t+1}^{\varepsilon-1} F_{t+1}.$$

Here π_t denotes the gross rate of inflation.

It is well known that aggregate output can be written as[7]

$$Y_t = p_t^* h_t \tag{2.10}$$

where p_t^* is a measure of price dispersion, which evolves according to

$$p_t^* = \left[(1-\theta) \left[\frac{1-\theta \pi_t^{\varepsilon-1}}{1-\theta} \right]^{\frac{-\varepsilon}{1-\varepsilon}} + \theta \pi_t^{\varepsilon} (p_{t-1}^*)^{-1} \right]^{-1}.$$

[7]See for example Woodford (2003).

The aggregate resource constraint is given by

$$C_t + G_t \leq Y_t. \tag{2.11}$$

In equilibrium, this constraint is satisfied as an equality because households and the government go to the boundary of their budget constraints. We consider two representations for the G_t process. In the first, the government is a constant, G, independent of r_t for all t. In the second

$$G_t = \begin{cases} G & r_t = r \\ G^\ell & r_t = r^\ell \end{cases}.$$

The monetary policy rule is given by

$$R_t = \max\{1, 1 + r + \alpha(\pi_t - 1)\}. \tag{2.12}$$

The max operator reflects the ZLB constraint on nominal interest rates and α is assumed to be larger than $1 + r$. As in BSGU, the latter assumption guarantees the existence of two steady states.

The economy begins at time 0 with $r_0 = r^\ell$ and p^*_{-1} equal to its value in the non-stochastic steady state associated with $r_t = r^\ell$.

2.2. Equilibrium

It is convenient to collect the equilibrium conditions of the model:

$$\begin{aligned}
\left[(1-\theta)\left[\frac{1-\theta\pi_t^{\varepsilon-1}}{1-\theta}\right]^{\frac{\varepsilon}{\varepsilon-1}}+\frac{\theta\pi_t^{\varepsilon}}{p_{t-1}^{*}}\right]^{-1}-p_t^{*}&=0 \\
\frac{1}{1+r_t}\max\left(1,1+r+\alpha\left(\pi_t-1\right)\right)E_t\frac{1}{Y_{t+1}-G_{t+1}}\frac{1}{\pi_{t+1}}-\frac{1}{Y_t-G_t}&=0 \\
\frac{Y_t}{Y_t-G_t}+\theta\frac{1}{1+r_t}E_t\pi_{t+1}^{\varepsilon-1}F_{t+1}-F_t&=0 \\
\chi\frac{Y_t^2}{p_t^{*}}+\theta\frac{1}{1+r_t}E_t\pi_{t+1}^{\varepsilon}F_{t+1}\left[\frac{1-\theta\pi_{t+1}^{\varepsilon-1}}{1-\theta}\right]^{\frac{1}{1-\varepsilon}}-F_t\left[\frac{1-\theta\pi_t^{\varepsilon-1}}{1-\theta}\right]^{\frac{1}{1-\varepsilon}}&=0.
\end{aligned} \tag{2.13}$$

At time t the model has two state variables, lagged price dispersion, p_{t-1}^{*}, and the current period realization of the discount rate, r_t. We define a *rational expectations equilibrium* as a set of functions $Y_t=\mathbf{Y}(p_{t-1}^{*},r_t)$, $\pi_t=\boldsymbol{\pi}(p_{t-1}^{*},r_t)$, $F_t=\mathbf{F}(p_{t-1}^{*},r_t)$, $p_t^{*}=\mathbf{p}^{*}(p_{t-1}^{*},r_t)$ that satisfy (2.13) for every p_{t-1}^{*} and r_t. Note that the equations in (2.13) hold with strict equality, so we are restricting ourselves to interior equilibria. Since the equilibrium functions depend only on p_{t-1}^{*} and r_t, we restrict ourselves to minimum-state-variable equilibria. We denote the set of equilibrium functions by

$$\mathbf{X}\left(p_{t-1}^{*},r_t\right)=\left[\begin{array}{cccc}\mathbf{Y}(p_{t-1}^{*},r_t) & \boldsymbol{\pi}(p_{t-1}^{*},r_t) & \mathbf{F}(p_{t-1}^{*},r_t) & \mathbf{p}^{*}(p_{t-1}^{*},r_t)\end{array}\right]. \tag{2.14}$$

We now describe our recursive procedure for computing equilibria. We first solve for $\mathbf{X}\left(p_{t-1}^{*},r_t\right)$ for $r_t=r$. We then use this solution to find $\mathbf{X}\left(p_{t-1}^{*},r_t\right)$ when $r_t=r^{\ell}$. In both cases $\mathbf{X}$ solves a particular fixed point problem.

Consider $r_t=r$. Given a function, $\tilde{\mathbf{X}}\left(p_t^{*},r\right)$, and a value of p_{t-1}^{*}, we use (2.13) to solve for Y_t, π_t, p_t^{*}, F_t. By varying p_{t-1}^{*} we obtain a new function, $\bar{\mathbf{X}}(p_{t-1}^{*},r)$. This procedure defines a mapping on the space of candidate equilibrium functions.

An equilibrium, $\mathbf{X}(p^*_{t-1}, r)$, is a fixed point of the mapping. Now consider $r_t = r^\ell$. Conditional on $\mathbf{X}(p^*_{t-1}, r)$, $\mathbf{X}\left(p^*_{t-1}, r^\ell\right)$ is a fixed point of another mapping defined by (2.13). Interiority of the equilibrium requires that non-negativity constraints are satisfied for all $0 < p^*_{t-1} < 1$.

We compute model equilibria using a standard finite element method. In particular, we approximate $\mathbf{X}(p^*_{t-1}, r)$ and $\mathbf{X}\left(p^*_{t-1}, r^\ell\right)$ by functions that are piecewise linear and continuous in p^*_{t-1} (see the Appendix for details). We find two functions, $\mathbf{X}(p^*_{t-1}, r)$. Corresponding to each function, we find two functions, $\mathbf{X}\left(p^*_{t-1}, r^\ell\right)$. So, in total we find four equilibria.

We now discuss our procedure for finding two equilibrium functions, $\mathbf{X}\left(p^*_{t-1}, r\right)$. It is convenient to define p^* :

$$p^* = \mathbf{p}^*(p^*, r).$$

A *steady-state equilibrium* is the 4-tuple:

$$p^*,\ Y \equiv \mathbf{Y}(p^*, r),\ \pi \equiv \boldsymbol{\pi}(p^*, r),\ F \equiv \mathbf{F}(p^*, r).$$

As in BSGU, there are two steady-state equilibria corresponding to the two solutions to the second equation in (2.13), evaluated in steady state:

$$\frac{1}{1+r} \max\left(1, 1 + r + \alpha\left(\pi - 1\right)\right) \frac{1}{\pi} = 1$$

The solutions are $\pi = 1$ and $\pi = 1/\left(1+r\right)$. In each case the rest of the equilibrium is computed trivially using the other equations in (2.13). We construct two functions, $\mathbf{X}\left(p^*_{t-1}, r\right)$, where each satisfies one of the two steady states. We find that these two functions are distinct in the sense that there is no transition path from one steady state to the other. We call the function $\mathbf{X}$ associated with $\pi = 1$ the *high inflation* *function.* We call the function $\mathbf{X}$ associated with $\pi = 1/(1+r)$ the *low inflation function.*

We now discuss our procedure for finding two equilibrium functions, $\mathbf{X}\left(p_{t-1}^{*}, r^{\ell}\right)$, conditional on a given $\mathbf{X}(p_{t-1}^{*}, r)$. It is convenient to define p_{ℓ}^{*} :

$$p_{\ell}^{*} = \mathbf{p}^{*}(p_{\ell}^{*}, r^{\ell}).$$

A *ZLB steady-state equilibrium* is the 4-tuple:

$$p_{\ell}^{*},\ Y_{\ell} \equiv \mathbf{Y}(p_{\ell}^{*}, r^{\ell}),\ \pi_{\ell} \equiv \boldsymbol{\pi}(p_{\ell}^{*}, r^{\ell}),\ F_{\ell} \equiv \mathbf{F}(p_{\ell}^{*}, r^{\ell}).$$

To compute the ZLB steady state we use the facts: (i) (2.13) consists of four equations in the four unknowns, Y_{ℓ}, π_{ℓ}, F_{ℓ}, p_{ℓ}^{*} and (ii) these equations can be collapsed into one equation in the one unknown, π_{ℓ} :

$$f(\pi_{\ell}) = 0. \tag{2.15}$$

To define f, solve the first equation in (2.13) for $p_{\ell}^{*} = p_{t-1}^{*} = p_{t}^{*}$ conditional on a given value π_{ℓ}. Then, use $\mathbf{X}(p_{\ell}^{*}, r)$ and the second equation in (2.13) to solve for Y_{ℓ}. The third equation in (2.13) can then be used to solve for F_{ℓ}. The value of $f(\pi_{\ell})$ is the expression on the left of the equality in the fourth equation of (2.13).

For every model parameterization that we consider and for both of the associated $\mathbf{X}\left(p_{t-1}^{*}, r\right)$'s, there are two values of π_{ℓ} which solve (2.15). It follows that there are two ZLB steady state equilibria. We construct two functions, $\mathbf{X}\left(p_{t-1}^{*}, r^{\ell}\right)$, each of which satisfies one of the two steady states. These two functions turn out to be distinct, regardless of $\mathbf{X}\left(p_{t-1}^{*}, r\right)$ in the sense that there is no transition path from one ZLB steady state to the other. We refer to the $\mathbf{X}\left(p_{t-1}^{*}, r^{\ell}\right)$ associated with the

Table 2.1: The Four Equilibria in Our Model

	$r_t = r^{\ell}$	$r_t = r$
$\mathbf{X}_H^H\left(p_{t-1}^*, r_t\right)$	ZLB high inflation function	high inflation function
$\mathbf{X}_L^H\left(p_{t-1}^*, r_t\right)$	ZLB low inflation function	high inflation function
$\mathbf{X}_H^L\left(p_{t-1}^*, r_t\right)$	ZLB high inflation function	low inflation function
$\mathbf{X}_L^L\left(p_{t-1}^*, r_t\right)$	ZLB low inflation function	low inflation function

higher ZLB steady-state equilibrium inflation as the *ZLB high inflation function*. We refer to the $\mathbf{X}\left(p_{t-1}^*, r^{\ell}\right)$ associated with the lower ZLB steady-state equilibrium inflation as the *ZLB low inflation function*.

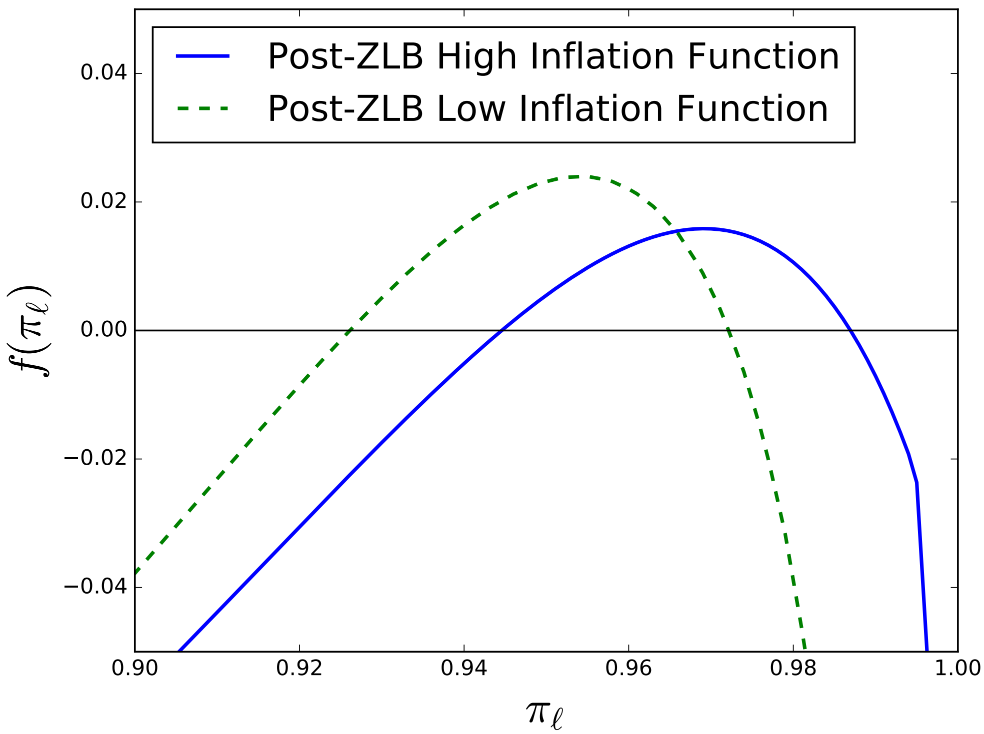
In sum, we have four equilibria: In practice, most of the literature focuses on the equilibrium obtained after linearizing the equilibrium conditions about the high inflation steady state. Most of our analysis focuses on the two equilibria, $\mathbf{X}_H^H\left(p_{t-1}^*, r_t\right)$ and $\mathbf{X}_L^H\left(p_{t-1}^*, r_t\right)$, in which the economy converges to the high inflation steady state after the ZLB is over. For simplicity, we refer to these as *post-ZLB high inflation equilibria*. Of these two equilibria, we refer to $\mathbf{X}_H^H\left(p_{t-1}^*, r_t\right)$ as the *bad ZLB equilibrium* and $\mathbf{X}_L^H\left(p_{t-1}^*, r_t\right)$ as the *really-bad ZLB equilibrium*. The results for the other equilibria are summarized in the text, and the associated formal analysis is contained in the appendix.

In our experiments we use following baseline parameterization of the model:

$$\begin{aligned} \varepsilon &= 7.0,\ \beta = 0.99,\ \alpha = 2.0,\ p = 0.75, \\ r^{\ell} &= -0.02/4,\ \theta = 0.85,\ G = 0.2,\ \chi = 1.25. \end{aligned} \tag{2.16}$$

In the remainder of this section, we display the equilibria corresponding to this parameterization. The two curves in Figure 2.1 display the function $f(\pi_{\ell})$ for two

Figure 2.1: Graph of $f\left(\pi_{\ell}\right)$



cases. The solid and dashed lines correspond to the post-ZLB high and low inflation equilibria, respectively. Notice that in each case $f(\pi_\ell)$ has an inverted 'U' shape and there are two ZLB steady state equilibria.

Figure 2.2 displays the four equilibrium functions,

$$p_t^* = \mathbf{p}^{*i}_j(p_{t-1}^*, r^\ell),\ i, j = H, L.$$

Here, $\mathbf{p}^*$ represents the fourth function in $\mathbf{X}$ (see (2.14)). The super and subscripts attached to $\mathbf{p}^*$ have the same interpretation as the those in Table 2.1. For convenience, Figure 2.2 also includes the 45 degree line. Each curve converges to the 45 degree line from above as p_{t-1}^* converges to zero.[8] Also, as p_{t-1}^* increases, each curve is strictly increasing and cuts the 45 line from above at exactly one point. In each case that point corresponds to the relevant ZLB steady state. So, p_t^* converges monotonically to the ZLB steady state (if the ZLB lasts long enough) for all $p_{t-1}^* > 0$. Finally, we assess the accuracy of our approximation to $\mathbf{X}$ by evaluating the equations in (2.13), i.e., the error functions, over a grid of values of p_{t-1}^* that is 10 times finer than the grid used in constructing our approximating functions. The maximal absolute value of the error function is 10^{-5}.

2.3. Baseline Results

In this section, we consider the following experiment. At time $t = -1$ the economy is in a nonstochastic steady state for $r_{-1} = r$ and $\pi_{-1} = p_{-1}^* = 1$. Consistent with (2.1), agents assign zero probability to a change in r_t. In period $t = 0$, $r_0 = r^\ell$. Afterward, r_t evolves according to (2.1). The dotted lines in each panel of Figure 2.3 display the dynamic path of inflation and consumption, respectively, after the shock to r_t

[8] That $p_t^* \to 0$ as $p_{t-1}^* \to 0$. This is obvious from the first equation in (2.13).

Figure 2.2: Graph of $p_t^* = \mathbf{p}^*(p_{t-1}^*, r^\ell)$

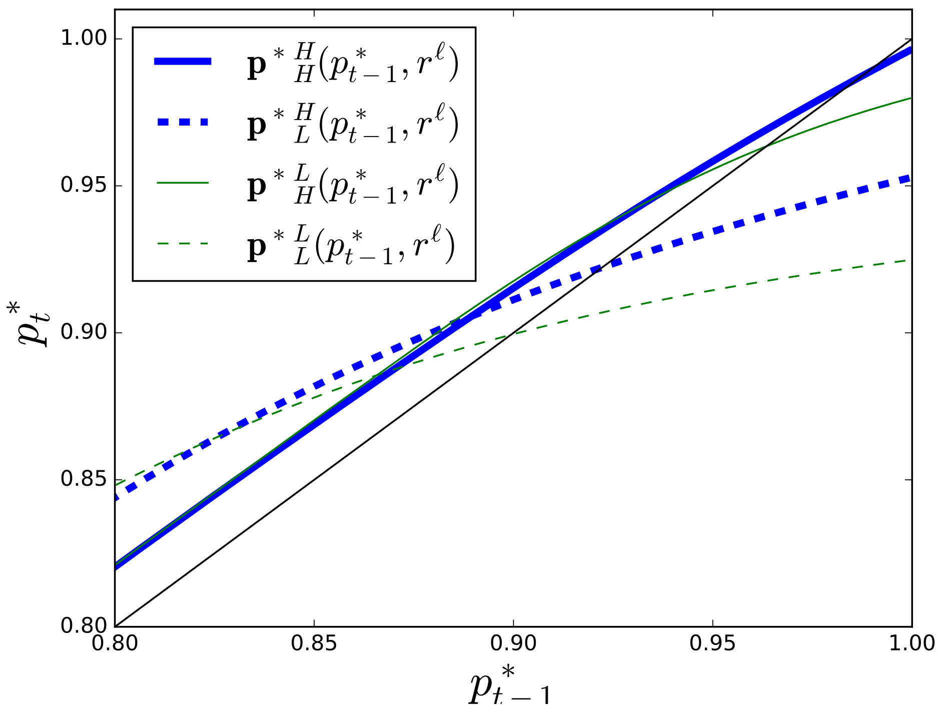


Figure 2.3: RE Equilibrium Paths In ZLB

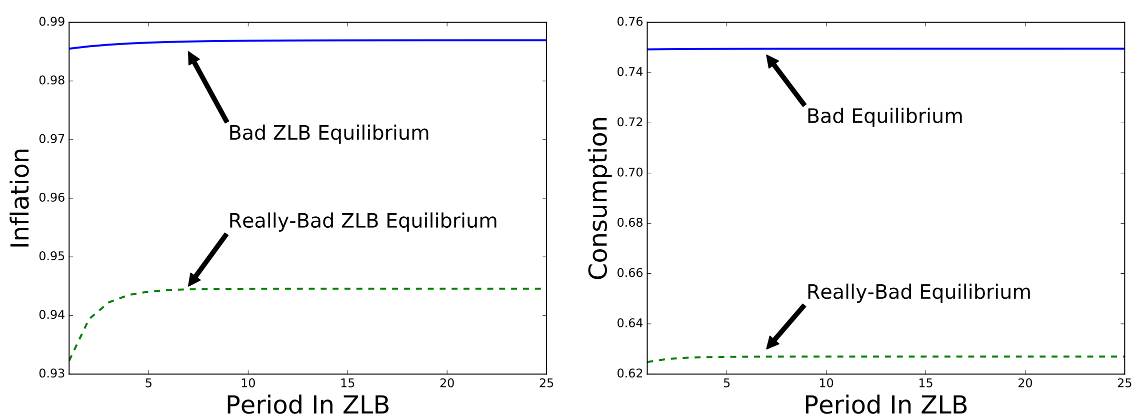
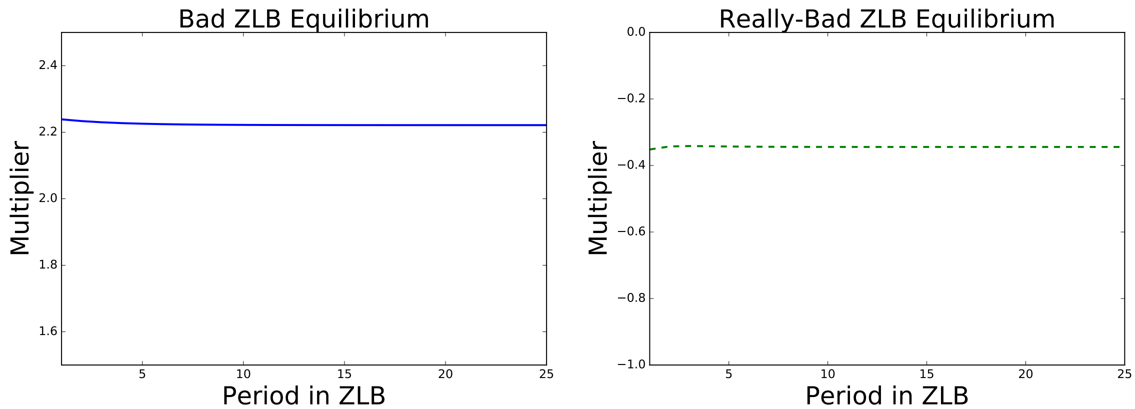


Figure 2.4: RE Multiplier at the ZLB



and as the economy converges to bad and really-bad ZLB steady state equilibria. These paths were computed by simulating $\mathbf{X}_H^H\left(p_{t-1}^*, r^\ell\right)$ and $\mathbf{X}_L^H\left(p_{t-1}^*, r^\ell\right)$ with an initial condition, $p_{-1}^* = 1$. A number of features are worth noting. First, in the bad equilibrium, quarterly inflation and consumption drop in the impact period of the shock by about 1.5 and 5.1 percentage points, respectively. After about 5 quarters these declines stabilize at 1.3 and 5 percentage points, respectively. Second, in the really-bad ZLB equilibrium, quarterly inflation and consumption drop in the impact period of the shock by 6.8 and 17.5 percentage points, respectively. After about 5 quarters these declines stabilize at 5.5 and 17.3 percentage points, respectively. Third, the dynamics induced by the evolving state variable p_t^* are larger in magnitude for the really-bad ZLB equilibrium. The system effectively converges after one year.

To derive values for the multiplier we assume that $G^\ell = 1.05 \times G^h$, i.e. when the economy is in the ZLB, G rises by 1 percent of steady state output. We define the multiplier in the first period to be

$$\frac{G^{\ell}}{(C^{\ell}(p_{t-1}^{*})+G^{\ell})}\frac{\Delta\left(C^{\ell}(p_{t-1}^{*})+G^{\ell}\right)}{\Delta G^{\ell}}.$$

We compute this ratio assuming that if the economy is in the bad (really-bad) ZLB equilibrium for a low value of G, it is in the bad (really-bad) ZLB equilibrium for the high value of G.[9] The two panels of Figure 2.4 display the multiplier in the bad and really-bad ZLB equilibrium as a function of time. Notice that the multiplier in the bad ZLB equilibrium is large, exceeding two over the time period displayed. In contrast, the multiplier is actually *negative* in the really-bad ZLB equilibrium. This change in sign is a dramatic illustration of the basic result in Mertens and Raven (2015) where the multiplier is much lower in the analog to our really-bad ZLB equilibrium. To understand why the sign of the multiplier depends on which equilibrium we are in, note that an increase in G^{ℓ} shifts f upwards (see Figure 2.5). This shift implies that the effect of an increase in G depends on which equilibrium we consider because of the inverted U shape of f.

The size of the multiplier in the high-inflation ZLB equilibrium increases as p rises or θ falls, i.e. as the expected duration of the ZLB rises or as prices become more flexible. These results are consistent with the intuition in CER (2011) and EW. In contrast, the size of the multiplier associated with the low-inflation ZLB equilibrium become more negative as the multiplier increases as p rises or θ falls.

[9] This assumption is non-trivial because one can easily construct examples in which G serves as a sunspot inducing a switch from one equilibrium to the other. As in Mertens and Raven (2015), we abstract from this issue.

Figure 2.5: RE Multiplier In ZLB

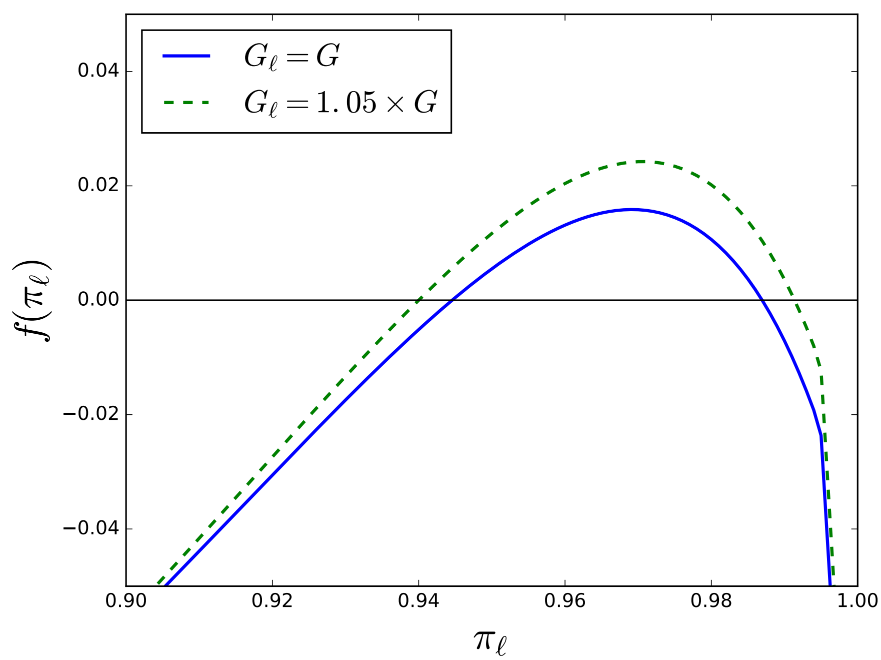


Table 2.2: Comparing the Linear and Non-linear Models

Model	Output	Inflation	Multiplier
Linear	-2.18	-0.0066	1.63
Nonlinear, high-inflation	-2.84	-0.0093	2.24
Nonlinear, low-inflation	-17.87	-0.0734	-0.35

2.3.1. Comparisons to linearized version of the model

Table 2.2 summarizes our results regarding the impact of changes in G for the non-linear and linear versions of the Calvo model. We report the responses of inflation, output and the multiplier in the impact period of a shock to the discount rate accompanied by a rise in G. Notice that the equilibrium behavior of the linearized model is similar to that of the non-linear model in the high-inflation ZLB equilibrium. For example, the impact multiplier in the linear model is 1.63 while it is 2.24 in the high-inflation ZLB equilibrium. While the magnitudes of the two multipliers are different, both deserve the adjective 'large'. The initial percent drop in GDP in the linear and high-inflation ZLB equilibrium model is 2.18% and 2.84% respectively. Again, while the numbers are different, the decline in output is large in both cases. In stark contrast, the properties of the non-linear model in the low-inflation ZLB equilibrium are very different than those of the linear model. For example the impact multiplier is -0.35 and the initial drop in GDP is 17.87%.

The multiplier in the linear model is inversely related to a term given by:

$$\Delta = (1-p)(1-\theta p) - p(2-G/Y)\frac{(1-\theta)(1-\beta\theta)}{\theta}.$$

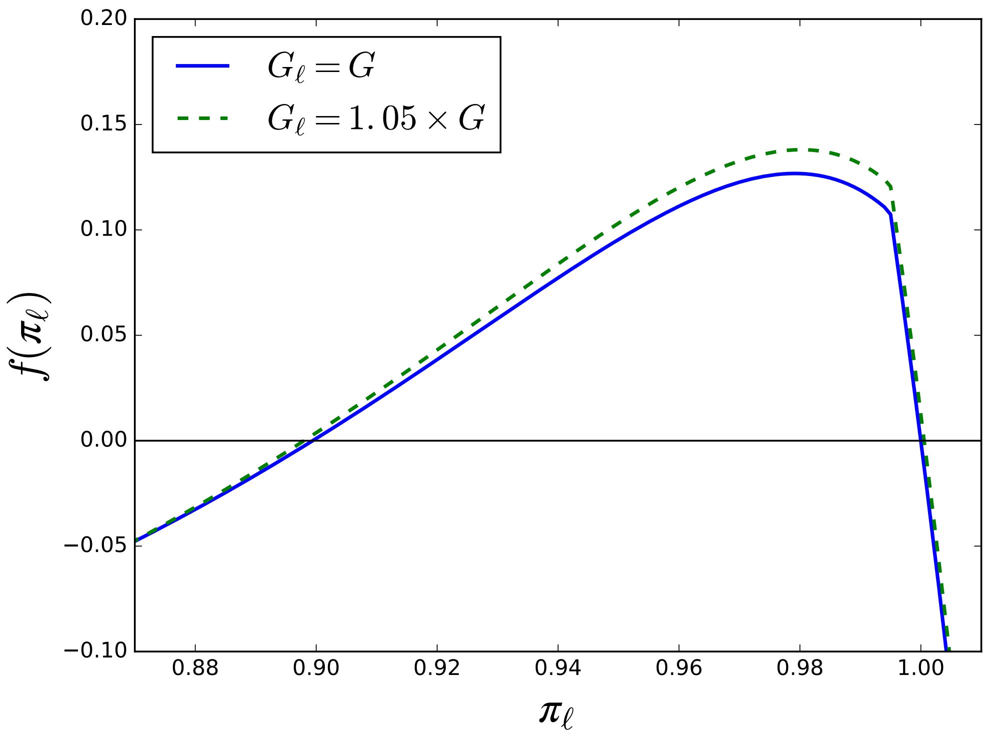
It is evident that the multiplier is strictly increasing in p and θ. See CER (2015) for the intuition underlying this result. As noted above, a similar result obtains for the high-inflation ZLB equilibria of the non linear model. There is one interesting difference between the linear and non- linear models. Carlstrom, Fuerst and Paustian (2014) prove that the linear model does not have an interior equilibrium when Δ is negative. Before Δ turns negative, the multiplier can be *arbitrarily* large. We found that increases in p and declines in θ, which reduce the value of Δ, have the effect of shifting the $f(\pi_\ell)$ function down. At some point $f(\pi_\ell)$ is not equal to zero for any π_ℓ, i.e. an interior equilibrium non longer exists. So non-existence leads to an effective bound on the multiplier in the non-linear model. In practice we found that the upper and lower bounds associated with bad and really-bad ZLB equilibria were 4.3 and -2.5 percent, respectively.

To summarize, the basic qualitative results reported in CER using a log-linear approximation obtain when we consider the nonlinear solution as long as we focus attention on the bad ZLB equilibrium.

2.4. Sunspot Equilibria

In the analysis above, we assumed that the ZLB becomes binding because of a shock to the household's discount rate. We now consider a scenario in which the ZLB binds because of a non-fundamental shock. This case is the one considered by Mertens and Ravn (2015). Suppose that at $t = 0$, before any agent has made a decision, the economy is in the high-inflation steady state equilibrium. Each firm observes a sunspot. Conditional on the sunspot, firms can either believe that other firms behave as in they did in the high inflation steady state or firms will set their prices sufficiently low to make the ZLB bind. With probability p firms continue to hold this belief. With probability $(1 - p)$, firms believe that other firms will set their prices sufficiently high to make the ZLB non-binding and behave as they did in the

Figure 2.6: Steady State Sunspot Equilibrium Function

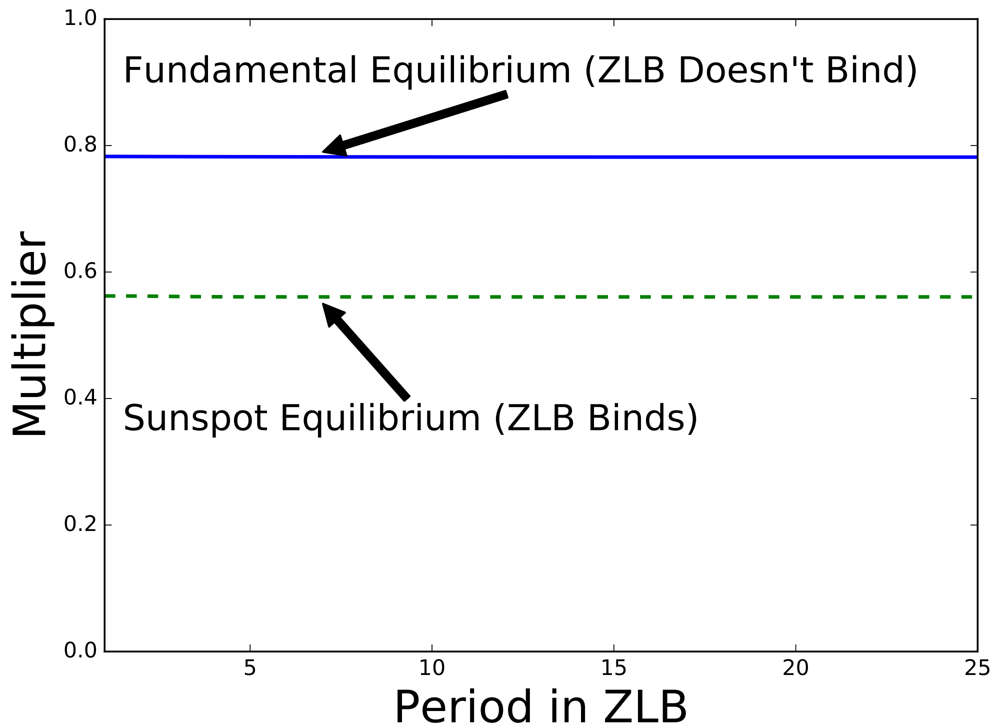


target-inflation steady state. The latter belief is an absorbing state.

Figure 2.6 displays the $f(\pi_\ell)$ function for the case under consideration. Notice that there are two steady state ZLB equilibria corresponding to a low and high inflation rate, respectively.

As stressed in Mertens and Ravn (2015), the sunspot equilibrium can be characterized as a situation in which the shock driving the economy into a binding ZLB scenario is a loss in confidence. The basic intuition is as follows. Suppose that agents anticipate deflation, creating the perception that the real interest rate is high. Households respond to the high real interest rate by reducing expenditures, thus driving the economy into a recession. The lower level of output leads to a fall in real

Figure 2.7: Sunspot Multipliers



wages and marginal cost. The latter effect leads to sustained downward pressure on the price level because of price-setting frictions. So the initial fear of deflation is self-fulfilling. Mertens and Ravn (2015) propose this non-fundamental 'loss of confidence' shock as an alternative to a fundamental shock that drives the economy into the ZLB.

Now consider the effect of a rise in G when the sunspot occurs. Note that the ZLB is only binding in one of the two sunspot equilibria. Figure 2.7 displays the multiplier as a function of time for the case where the economy is in the ZLB and the case where the interest rate remains above zero. As it turns out, the multiplier at the ZLB can be larger or smaller than in the steady state, depending on parameter values. The robust result is that the multiplier is quite small: (0.56) in the ZLB and (0.79) away from the ZLB. The multiplier is small when the ZLB is not binding because an increase in government spending leads to inflation. Monetary policy responds by raising the real interest rate which crowds out private consumption. In the equilibrium at the ZLB, the real interest rate is also high because the ZLB binds and there is deflation. So again consumption falls, in this case because inflation falls, leading to a relatively small multiplier.

3. Stability Under Learning at the ZLB

In this section we investigate stability under learning of the bad and really-bad ZLB equilibria. In order to determine what happens when agents don't have rational expectations, we must make assumptions about how their beliefs evolve over time.

3.1. The benchmark case

In the rational expectations version of the Calvo model, intermediate good firms choose their price level, $\tilde{P}_{j,t}$, based in part on the value of the aggregate price level, P_t. But, the latter is a function of firms' collective pricing decisions. So firms cannot actually observe P_t when they choose $\tilde{P}_{j,t}$. The standard assumption is that these firms form a 'belief' about P_t, based on state variables and contemporaneous shocks, when they make their decisions. In a rational expectations equilibrium that belief is correct. In a world where firms don't necessarily have rational expectations it is not natural to assume that firms actually see P_t at the time they choose $\tilde{P}_{j,t}$. Note that if firms don't see P_t they also don't know the demand for their output.

At time t firms make their decisions given the state variables p^*_{t-1} and r_t, as well as views about the equilibrium values for current and future values of P_t and C_t. We assume that firms believe they are in a stationary environment. That is, firms think that the mapping from states and shocks to equilibrium values of aggregate prices and quantities won't change over time. Denote by $\mathbf{X}^{e,f}(p^*_{t-1}, r^\ell, t-1)$ the typical firm's belief, formed using information up to time $t-1$, about the equilibrium function while $r_t = r^\ell$. While the firm knows the actual value of p^*_{t-1}, we must attribute to it beliefs about the entire equilibrium function for $\mathbf{X}$. The reason is that the firm's first order conditions involve equilibrium prices and quantities in the current period and in future periods. We include time $t-1$ as an argument to $\mathbf{X}^{e,f}$ in order to capture the assumption that firms think they are in a stationary environment, i.e. they don't expect that their beliefs about the functions will change in the future (see Evans and Honkapohja (2001)). However, over time beliefs do evolve.

Given new information, firms' beliefs evolve over time according to

$$\mathbf{X}^{e,f}(\cdot, r^\ell, t) = \omega \mathbf{X}(\cdot, r^\ell, t-1) + (1-\omega)\mathbf{X}^{e,f}(\cdot, r^\ell, t-1). \tag{3.1}$$

Here, $\mathbf{X}(\cdot, r^\ell, t-1)$ is the mapping from p^*_{t-1} to equilibrium prices and quantities, given beliefs at time $t-1$. For $\omega > 0$, this formulation assumes that at time t, agents know $\mathbf{X}(\cdot, r^\ell, t-1)$. This assumption is clearly heroic. So we also investigate what happens when firms just assume that the values of the variables that they have to forecast are equal to their current values (see the appendix for details). As an aside, it is worth noting that in Rotemberg model, discussed in Section 5, there are no state variables other than r_t. So we can replace (3.1) with the assumption that agents' expectations about the values of future variables evolve according to a simple constant gain algorithm.

When households make their time t consumption decisions, firms' actions have already determined the aggregate price level. Given this information, the households can compute the time t equilibrium function for inflation.[10] Denote by $\mathbf{X}^{e,h}(p^*_{t-1}, r^\ell, t)$ households' belief, at time t, about the equilibrium mapping from states and shocks to equilibrium prices and quantities. Households think they are in a stationary environment, i.e. they don't expect that their beliefs about this function will change in the future. Given new information, households beliefs evolve according to

$$\mathbf{X}^{e,h}(\cdot, r^\ell, t) = \omega \mathbf{X}(\cdot, r^\ell, t-1) + (1-\omega)\mathbf{X}^{e,h}(\cdot, r^\ell, t-1). \tag{3.2}$$

When $r_t = r_\ell$, firms' optimality conditions can be written as

$$\frac{\tilde{P}_{\ell,t}}{P_{t-1}} = \frac{P_t}{P_{t-1}} \frac{K^{e,f}_{\ell,t}}{F^{e,f}_{\ell,t}} \tag{3.3}$$

where

$$K^{e,f}_{\ell,t} = \chi \frac{(Y^{e,f}_{\ell,t})^2}{p^*_t} + \theta \frac{1}{1+r_\ell} (\pi^{e,f}_{\ell,t+1})^{\varepsilon} \left[p K^{e,f}_{\ell,t+1} + (1-p) K^{e,f}_{n,t+1} \right] \tag{3.4}$$

and

$$F^{e,f}_{\ell,t} = \frac{Y^{e,f}_{\ell,t}}{C^{e,f}_{\ell,t}} + \theta \frac{1}{1+r_\ell} (\pi^{e,f}_{\ell,t+1})^{\varepsilon - 1} \left[p F^{e,f}_{\ell,t+1} + (1-p) F^{e,f}_{n,t+1} \right]. \tag{3.5}$$

Here, $F^{e,f}_{\ell,t}$, $K^{e,f}_{\ell,t}$, $Y^{e,f}_{\ell,t}$, $C^{e,f}_{\ell,t}$, and $\pi^{e,f}_{\ell,t}$ denote firms' beliefs about F_t, K_t, Y_t, C_t, and π_t when $r_t = r^\ell$. Similarly, $F^{e,f}_{n,t}$, $K^{e,f}_{n,t}$, $Y^{e,f}_{n,t}$, $C^{e,f}_{n,t}$, and $\pi^{e,f}_{n,t}$ denote firms' beliefs about F_t, K_t, Y_t, C_t, and π_t when $r_t = r$. Note that the beliefs $F^{e,f}_{\ell,t}$, $K^{e,f}_{\ell,t}$, and $Y^{e,f}_{\ell,t}$ are derived from the beliefs $C^{e,f}_{\ell,t}$, and $\pi^{e,f}_{\ell,t}$ which evolve according to (3.1).

The first-order conditions of the household when $r_t = r_\ell$ can be written

$$\chi C_{\ell,t} h_{\ell,t} = \frac{W_{\ell,t}}{P_{\ell,t}}, \tag{3.6}$$

[10] They can do so under the further heroic assumption that they can solve the problem that the firm just solved.

and

$$\frac{1}{C_{\ell,t}} = \frac{1}{1+r_\ell} \max\{1, 1+r+\alpha(\pi_{\ell,t}-1)\} \left[\frac{p}{C^{e,h}_{\ell,t+1}\pi^{e,h}_{\ell,t+1}} + \frac{1-p}{C^{e,h}_{n,t+1}\pi^{e,h}_{n,t+1}} \right]. \tag{3.7}$$

Here $C_{\ell,t}$, $h_{\ell,t}$, $R_{\ell,t}$, and $\frac{W_{\ell,t}}{P_{\ell,t}}$ are the time t realized values of consumption, labor supply, the nominal interest rate, and the real wage. Household beliefs about future inflation evolve according to (3.2).

Definition 1. *A learning ZLB equilibrium is a sequence of functions $\mathbf{X}(\cdot, r^\ell, t)$ that satisfy the resource constraint, the monetary policy rule, and the household and firm optimality conditions for all t, given an initial set of beliefs $\mathbf{X}^{e,f}(\cdot, r^\ell, 0)$ and $\mathbf{X}^{e,h}(\cdot, r^\ell, 0)$, that evolve according to (3.1) and (3.2).*

Here we have assumed that households and firms know the equilibrium functions when $r_t = r$, i.e. they have rational expectations about the economy when it's not in the ZLB.

Our selection criterion for a rational expectations equilibrium is based on the following notion of stability.

Definition 2. *Suppose that in a neighborhood of a rational expectations equilibrium, either $\mathbf{X}^{e,f}(\cdot, r^\ell, 0)$ is not equal to $\mathbf{X}(\cdot, r^\ell)$ or $\mathbf{X}^{e,h}(\cdot, r^\ell, 0)$ is not equal to $\mathbf{X}(\cdot, r^\ell)$. Here $\mathbf{X}(\cdot, r^\ell)$ is the rational expectations equilibrium mapping from states and shocks to equilibrium prices and quantities. A rational expectations equilibrium is said to be stable-under learning if $\lim_{t\to\infty} \mathbf{X}(\cdot, r^\ell, t) \to \mathbf{X}(\cdot, r^\ell)$. That is, if the learning equilibrium converges to the rational expectations equilibrium.*

If the economy stays in the ZLB forever, it will converge to a ZLB steady state equilibrium. From our definition, if the rational expectations equilibrium is stable under learning, a learning equilibrium in the neighborhood of a rational expectations equilibrium must also approach the same ZLB steady state. This fact is very useful because it allows us to eliminate *all* equilibria that lead to the really-bad ZLB equilibrium as not being stable-under-learning.

We now establish this numerically and provide the underlying intuition. To this end, suppose that a firm incorrectly believes that the ZLB steady state inflation rate is $\pi_\ell^{e,f}$ and that the economy is in the corresponding ZLB steady state. Also, assume that p_{t-1}^* is consistent with this belief. Since the belief $\pi_\ell^{e,f}$ is not a rational expectations belief, $f(\pi_\ell^{e,f})$ is not equal to zero. Note that there is an equivalence between the belief $\pi_\ell^{e,f}$ and a belief about the value of $\tilde{p}_\ell^{e,f}$ that will be chosen by firms who can update their price. We use the function $f(\pi_\ell^{e,f})$ to define a new function $\tilde{f}\left(\tilde{p}_\ell^{e,f}\right)$ that must be equal to zero at a ZLB steady state equilibrium. Combining (3.3)-(3.5), and using the aggregate resource (2.11) and the household Euler equation (3.7), we represent the first-order condition of the firm under consideration as

$$\tilde{F}\left(\tilde{p}_\ell, \tilde{p}_\ell^{e,f}\right) = 0. \tag{3.8}$$

Define the best-response function

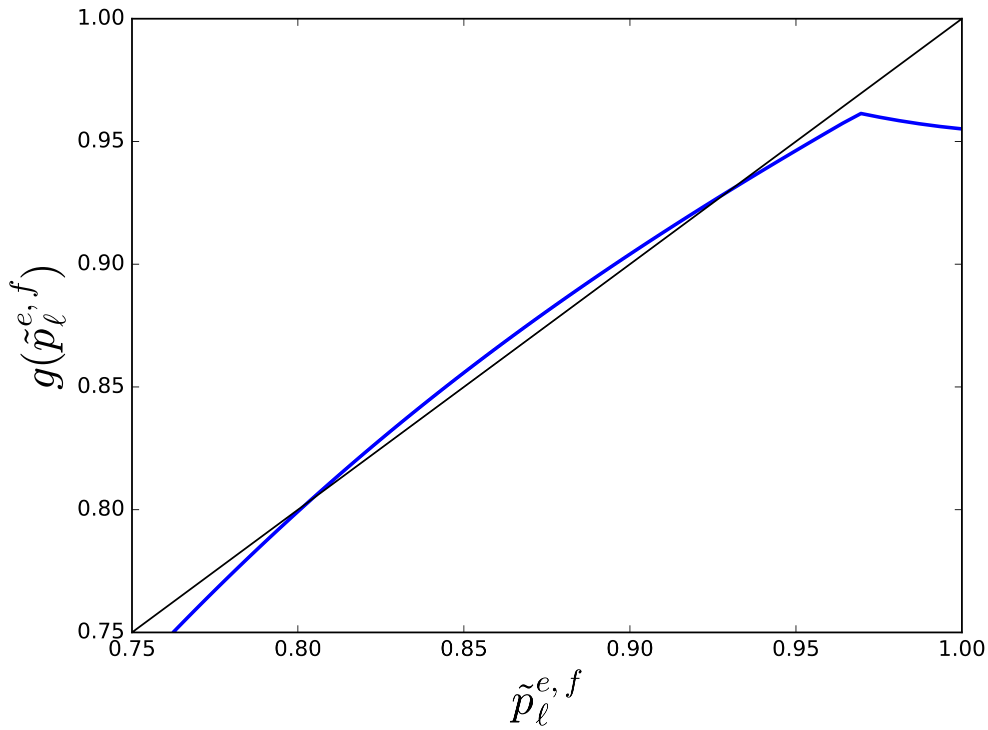
$$\tilde{p}_\ell = g(\tilde{p}_\ell^{e,f}). \tag{3.9}$$

This function has the property,

$$\tilde{F}\left(g(\tilde{p}_\ell^{e,f}), \tilde{p}_\ell^{e,f}\right) = 0. \tag{3.10}$$

That is, for arbitrary $\tilde{p}_\ell^{e,f}$, equation (3.8) is satisfied. In a ZLB steady state equilib-

Figure 3.1: Best Response Function



rium

$$\tilde{p}_\ell = \tilde{p}_\ell^{e,f}. \tag{3.11}$$

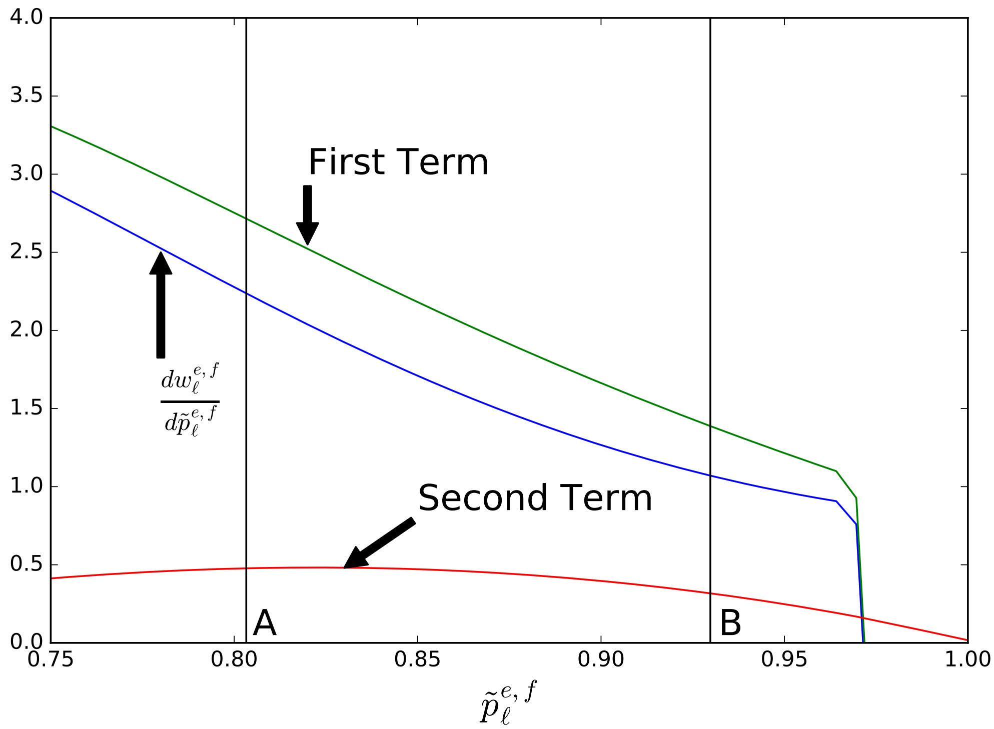
Figure 3.1 plots the typical firm's best response function (3.9). The two ZLB steady state equilibria correspond to the two points where the best response function intersects the 45 degree line. Notice that given any belief, $\tilde{p}_\ell^{e,f}$, between the rational expectations beliefs, the best response $g(\tilde{p}_\ell^{e,f})$ is greater than $\tilde{p}_\ell^{e,f}$. It follows that realized inflation will exceed beliefs about inflation. So, the learning equilibrium will move to the right, and toward the ZLB steady state equilibrium with higher inflation. Now consider any belief, $\tilde{p}_\ell^{e,f}$, that exceeds its value in the ZLB steady state equilibrium with higher inflation. Here the best response function $g(\tilde{p}_\ell^{e,f})$ is less than $\tilde{p}_\ell^{e,f}$. So realized inflation will be lower than beliefs about inflation and the learning equilibrium will move to the left, and toward the ZLB steady state equilibrium with higher inflation. Finally, consider any belief, $\tilde{p}_\ell^{e,f}$, that is less than its value in the ZLB steady state with low inflation. Here the best response function $g(\tilde{p}_\ell^{e,f})$ is less than $\tilde{p}_\ell^{e,f}$. It follows that realized inflation will be lower than beliefs about inflation. So the learning equilibrium will move away from the ZLB steady state equilibrium with low inflation.

The basic intuition for the previous results is as follows. Consider a firm whose expectations $\tilde{p}_\ell^{e,f}$ aren't equal to a value taken at a ZLB steady state equilibrium. Associated with $\tilde{p}_\ell^{e,f}$ is an expectation about aggregate consumption and the wage rate. In the ZLB, a low value of $\tilde{p}_\ell^{e,f}$ implies a low expected value of inflation and a high value of the real interest rate. From the household's Euler equation, a high real rate means that aggregate consumption will be low. The production function and the household's first-order condition imply that aggregate employment and the real wage will also be low. It follows that a low value of $\tilde{p}_\ell^{e,f}$ is associated with a low expected value of marginal cost. Since the firm's price is an increasing function of marginal cost, a low value of $\tilde{p}_\ell^{e,f}$ will be associated with a low value of $\tilde{p}_\ell$. This result is shown in Figure 3.1 since the best response function is an increasing function of $\tilde{p}_\ell$ when the ZLB binds.

We now show that a small change in $\tilde{p}_\ell^{e,f}$ is associated with a smaller movement in marginal cost when we start from the bad ZLB equilibrium (point B in Figure 3.2) than when we start from the really-bad ZLB equilibrium (point A in Figure 3.2). The basic force driving the result is that the consumption response to changes in inflation are much larger when inflation is low (near point A) than when inflation is high (near point B).

Consider the change in the expected real wage associated with a change in $\tilde{p}_\ell^{e,f}$:

Figure 3.2: Expected Changes in Marginal Cost



$$\frac{dw_t^{e,f}}{d\tilde{p}_\ell^{e,f}} = \frac{d\left(\chi\frac{(G_\ell^{e,f}+C_\ell^{e,f})C_\ell^{e,f}}{p_\ell^*}\right)}{d\tilde{p}_\ell^{e,f}} = \chi\left(\frac{G_\ell^{e,f}+2C_\ell^{e,f}}{p_\ell^*}\right)\frac{dC_\ell^{e,f}}{d\pi_\ell^{e,f}}\frac{d\pi_\ell^{e,f}}{d\tilde{p}_\ell^{e,f}} - \chi\left(\frac{(G_\ell^{e,f}+C_\ell^{e,f})C_\ell^{e,f}}{(p_\ell^*)^2}\right)\frac{dp_\ell^*}{d\tilde{p}_\ell^{e,f}}. \tag{3.12}$$

Figure 3.2 displays the behavior of this derivative as a function of $\tilde{p}_\ell^{e,f}$. As it turns out, the key determinant of this derivative is the first group of terms on the right hand side of the equation. That term captures the effects of $\tilde{p}_\ell^{e,f}$ on aggregate consumption, hours worked, and the real wage that operate through the real interest rate. To analyze this effect, we re-write the household's Euler Equation when the ZLB binds as

$$1 = \frac{1}{1+r_\ell}\left[\frac{p}{\pi_\ell} + \frac{(1-p)C_\ell}{C(p_\ell^*)\pi(p_\ell^*)}\right]. \tag{3.13}$$

When π_ℓ is near 1, there is a negative relationship between C_ℓ and π_ℓ that is roughly linear. For values of π_ℓ that are relatively far from 1, C_ℓ is more sensitive to changes in π_ℓ. This increased sensitivity reflects the convexity of the term p/π_ℓ which appears in the household's Euler equation. Since $\frac{d\pi_\ell^{e,f}}{d\tilde{p}_\ell^{e,f}}$ is roughly a constant, this convexity implies that the derivative of wages with respect to $\tilde{p}_\ell^{e,f}$ is much larger at point A than at point B.

Next, consider the second term in (3.12) that involves $\frac{dp_\ell^*}{d\tilde{p}_\ell^{e,f}}$. This term captures the impact of changes in p_ℓ^* on marginal cost induced by a change in $\tilde{p}_\ell^{e,f}$. Equation (2.10) implies that the amount of labor required to produce a given Y_t depends negatively on p_t^*. At the ZLB steady state, an increase in $\tilde{p}_\ell^{e,f}$ leads to a rise in inflation and a higher value of p_ℓ^*. So less labor is needed to produce the same amount of output. This effect induces a decline in hours worked, the real wage rate, and marginal cost.

Figure 3.2 displays $\frac{dp^*_\ell}{d\tilde{p}^{e,f}_\ell}$ as a function $\tilde{p}^{e,f}_\ell$.[11] Note that at point B, p^*_ℓ is near one and $\frac{dp^*_\ell}{d\tilde{p}^{e,f}_\ell}$ is near zero. In contrast, at point A, p^*_ℓ is roughly 0.9 and $\frac{dp^*_\ell}{d\tilde{p}^{e,f}_\ell}$ is greater than zero. A firm contemplating a decrease in $\tilde{p}^{e,f}_\ell$ thinks that marginal costs are falling more if its initial expectations are near point A rather than point B. But this effect is small relative to the impact of the first term in (3.12).

Critically, $\frac{dw^{e,f}_t}{d\tilde{p}^{e,f}_\ell}$ is bigger at point A than at point B. So at point A a firm will increases its price by *more* than it would at point B. Existence of equilibrium requires that the best response function crosses the 45 degree line at some point. Given that firms will increase their price by more the lower is $\tilde{p}^{e,f}$ in response to a change in beliefs about $\tilde{p}^{e,f}$, the equilibrium associated with point A has the property that a firm increases its price by more than one-for-one with an increase in $\tilde{p}^{e,f}_\ell$. In sharp contrast, a firm at point B will increase its price by *less* than one-for-one with expected increase in $\tilde{p}_\ell$. This result is precisely why the ZLB steady state equilibrium with relatively low inflation is not stable under learning and the ZLB steady state equilibrium with relatively high inflation is stable under learning.

To be stable-under-learning, the functions defining a learning equilibrium must converge point-wise to the functions defining a post-ZLB high-inflation equilibrium for every possible value of p^*_t. The previous discussion establishes that any equilibrium that converges to the ZLB steady state with relatively low inflation does not satisfy this condition, and is therefore not stable-under-learning. It does not establish that an equilibrium that converges to other ZLB steady state is stable-under-learning. We now establish, numerically, the stability-under-learning of the such an equilibrium.

Recall that we approximate the equilibrium functions $(\mathbf{X}(\cdot, r^\ell, t))$ with a finite

[11] We compute this term using the fact that $p^*_t = \left[(1-\theta)\tilde{p}_t + \theta\pi^\varepsilon_t(p^*_{t-1})^{-1}\right]^{-1}$.

elements method. That is, we approximate $\mathbf{X}(\cdot, r^\ell, t)$ with a finite number of parameters, z_t. The learning algorithm specified above defines a mapping from the current values of those parameters to the values that they take in the subsequent period

$$z_{t+1} = s\left(z_t\right). \tag{3.14}$$

Define

$$S(\tilde{z}) = \left[\frac{ds_i\left(z\right)}{dz_j}\right]|_{\tilde{z}}, \tag{3.15}$$

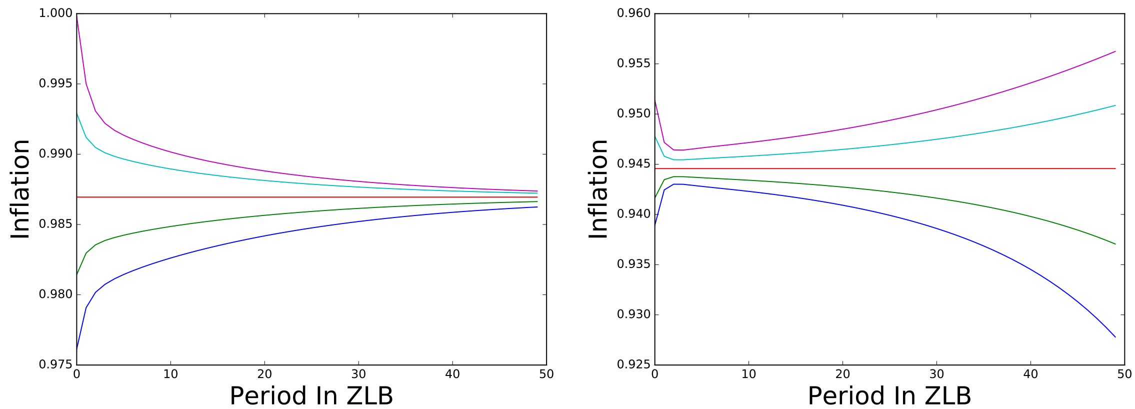
for all $i, j < N$ where N is the number of parameters. When we evaluate S for the parameters of the bad ZLB equilibrium, we find that the maximum eigenvalue is less than one in absolute value. This result establishes that for beliefs in a neighborhood of the bad ZLB equilibrium, the learning equilibrium will converge to the rational expectations equilibrium and the beliefs of households and firms will converge to the rational expectations beliefs. By contrast, when we evaluate S for the parameters of the really-bad ZLB equilibrium, we find that the maximum eigenvalue is greater than one in absolute value. This result implies that for initial beliefs in the neighborhood of the really-bad ZLB equilibrium, the learning equilibrium will diverge from the really-bad ZLB equilibrium, and beliefs will also diverge from the rational expectations beliefs.

To illustrate the process of convergence and divergence, suppose that at time -1 $p^*_{-1} = 1$. Then at time 0, $r_t = r^\ell$ and, for reasons unexplained, $\mathbf{X}^{e,f}(p^*_{-1}, r^\ell, -1) = \mathbf{X}(p^*_{-1}, r^\ell) + \bar{x}_\ell$, where $\bar{x}_\ell$ is a positive constant.[12] For simplicity we assume that the parameter ω in (3.1) and (3.2) is equal to one.

The first panel of Figure 3.3 displays the evolution of realized inflation for $\bar{x}_\ell=$

[12]We obtain virtually identical results regardless of whether $\bar{x}_\ell$ is applied to firms' beliefs about only inflation, only consumption, or both.

Figure 3.3: Learning Equilibria Near Steady State ZLB



(-.02, -.01, 0, 0.01, 0.02). The red line corresponding to $\bar{x}_\ell = 0.0$ is the inflation rate in the steady state of the bad ZLB equilibrium. Regardless of the value of $\bar{x}_\ell$, inflation converges to the steady state of the bad ZLB equilibrium. The second panel is the analog to the first, where we begin from the really-bad ZLB equilibrium. Here inflation diverges from that equilibrium in the learning equilibrium. For positive values of $\bar{x}_\ell$, inflation converges to the bad ZLB equilibrium. Interestingly, for $\bar{x}_\ell <$ 0, there does not exist an interior learning equilibrium with an infinite horizon as inflation eventually becomes so low that consumption is driven to zero.

Until now we have analyzed post-ZLB high-inflation equilibria. Now, we consider post-ZLB low-inflation equilibria. Figure 2.1 shows $f(\pi_\ell)$ for both cases. Notice that the curve is shifted to the left for the post-ZLB low-inflation equilibria, meaning that there are two ZLB steady states, and their inflation rates are lower than in the corresponding post-ZLB high-inflation equilibria. The reason that the curve is shifted to the left is that agents expect a lower rate of inflation after the ZLB is over. This effect means that the real interest in the ZLB is higher which leads to lower consumption.

It is still the case that analogue of the bad (really-bad) ZLB equilibrium is (not) stable-under-learning. So regardless of which assumption we make about agents beliefs about the post ZLB period, the bad ZLB equilibrium is stable-under-learning and the really-bad ZLB equilibrium is not.

4. Fiscal Policy in the Learning Equilibrium

In this section we analyze the value of government spending multipliers in learning equilibria.

4.1. Fiscal policy under benchmark learning scheme

We initially assume that agents think that when the ZLB episode is over, the economy reverts to post-ZLB high-inflation rational expectations equilibrium. Later we assess the robustness of our results to this assumption.

Assume that the economy begins in steady state, with $\pi_t = 1$. At time 0, $p^*_{-1} = 1$, and r_t falls to r^ℓ and evolves according to (2.1). Firms and households beliefs about equilibrium functions evolve according to (3.1) and (3.2).

Figure 4.1 displays the paths of consumption, inflation, and the government spending multiplier in the learning equilibrium (the blue lines), as well as the bad ZLB equilibrium paths (the green lines). The paths for inflation and consumption are computed holding government consumption at its steady state value (0.20). Notice that consumption and inflation converge to the bad ZLB steady state from above. The reason is that agents initially have expectations about inflation and consumption that are higher than warranted after the shock to r_t and expectations about higher future inflation and consumption spur demand in the present. As expectations ad-

Figure 4.1: Learning Equilibrium, Starting from Steady State

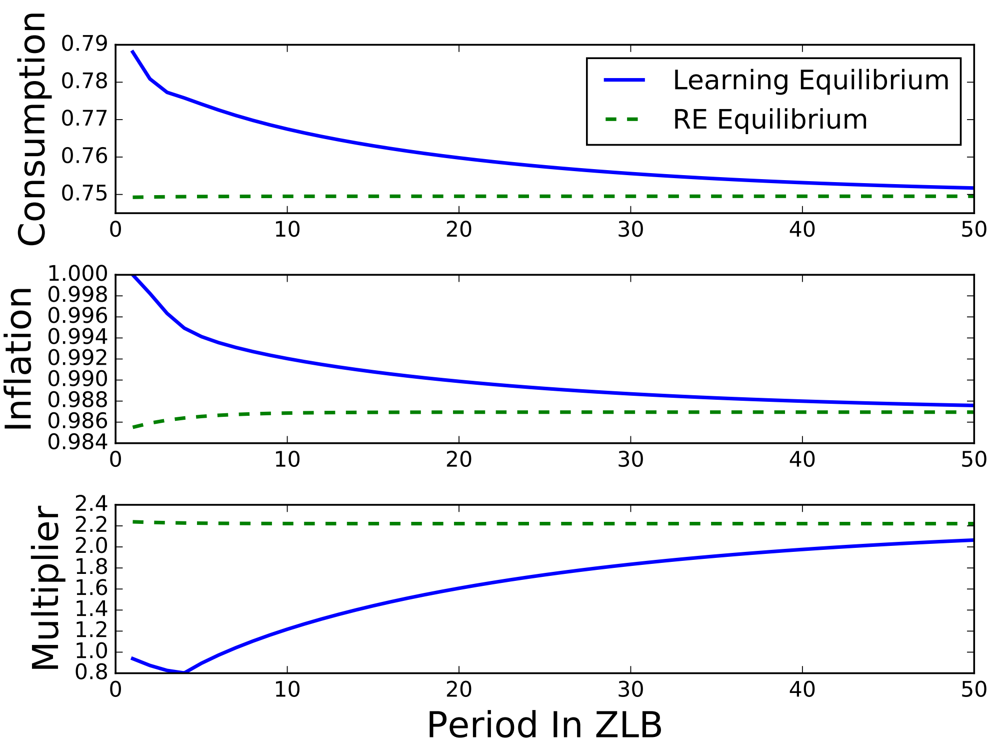
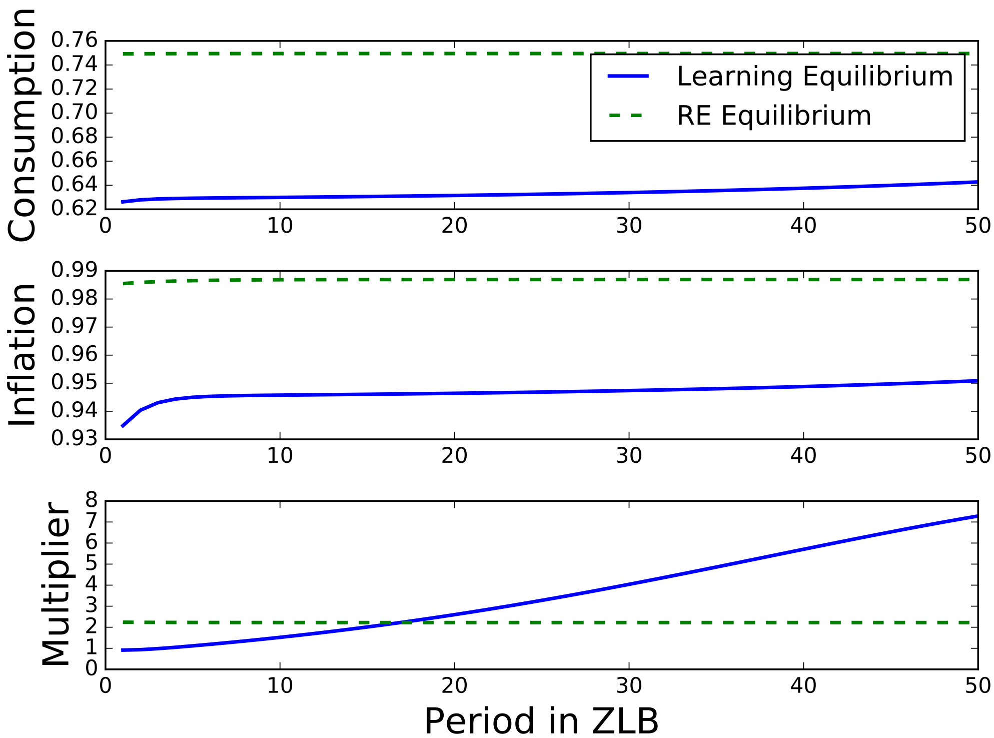


Figure 4.2: Learning Equilibrium, Starting Near Low-Inflation RE ZLB Equilibrium



just downward in response to the low realized inflation, inflation and consumption decline further. The value of the multiplier is initially low because the ZLB isn't binding in the first few periods. Once the ZLB starts to bind, the multiplier quickly rises above 1.

Now imagine that after the shock to r_t, firms and household have beliefs near the really-bad ZLB equilibrium. Figure 4.2 displays the paths of consumption, inflation, and the government spending multiplier in the learning equilibrium (the blue lines), as well as the bad ZLB equilibrium paths (the green lines). As before, the paths for inflation and consumption are computed holding government consumption at its steady state value (0.20). Notice that consumption and inflation converge to the bad

ZLB steady state from below. The reason is that agents initially expect future inflation and consumption to be low. These expectations reduce current consumption and inflation. The multiplier has an initial value of about 1 and then rises. The reason the multiplier rises is that the fiscal expansion helps quickly move expectations toward those associated with the bad ZLB equilibrium. Notably, the multiplier continues to rise for some time. After many periods, the multiplier eventually approaches the bad ZLB steady-state.

We conclude that in both scenarios the multiplier is large and eventually converges to its value in the bad ZLB equilibrium.

4.2. Reconciling with Mertens and Ravn (2015)

Mertens and Ravn (2015) report that the fiscal multiplier is small when they analyze a learning equilibrium near a rational expectations equilibrium that has similar properties to the really-bad ZLB equilibrium we consider. This result contrasts sharply with our result that the multiplier is very large when we begin near that equilibrium. There are four differences our analysis and theirs. First, they work with a linearized Calvo model when they study the learning equilibrium,. Second, they assume that firms who choose prices at time t see the time t aggregate price level when they choose prices. Third, they model household learning behavior about future consumption as in Evans and Honkapohja (2001). In contrast we suppose that households believe that the function mapping the state p_{t-1}^{*} to the household consumption decision is the same in the subsequent period. Fourth, the experiment that underlies their multiplier calculation is subtly but significantly different than ours. When we calculate the multiplier we initially consider an economy in which agents' initial expectations about inflation differ by ϵ_{π} from the really bad ZLB equilibrium. We then consider a separate economy with shocks that set r to r_ℓ and a shock to G. Agents' expectations about inflation and consumption start in the same place for the two economies. We then use the difference in output between the two economies to calculate the multiplier. Mertens and Ravn (2015) proceed in the same way with one crucial difference. When G increases, the rate of inflation in the really-bad ZLB equilibrium falls by ϵ'_π. When Mertens and Ravn raise G in the learning equilibrium, they also decrease agents' expectations about inflation by ϵ'_π. As discussed above, this fall in inflation expections would in and of itself reduce output in the ZLB.

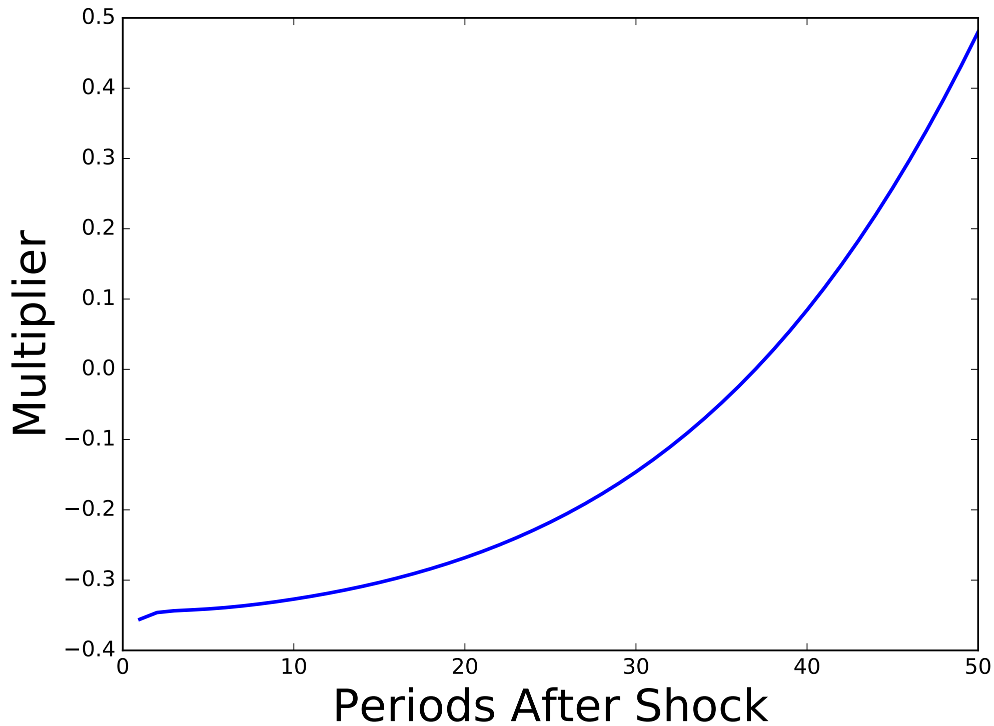
In the Appendix we show that first three differences between our analysis and Mertens and Ravn (2015) do not have a large impact on the multiplier calculations. In contrast the fourth difference is very important. Figure 4.3 displays the multiplier as a function of time if we adopt the assumption of Mertens and Ravn (2015) about how expectations about inflation change when G increases. Notice that we obtain a *negative* multiplier that persists for roughly 10 years. This results reflects that, in this example, the change in expectations is quantitatively much more important than the increase in G. From our perspective, the Mertens and Ravn experiment confounds the effects of two shocks.

5. The Rotemberg Model

A number of authors have studied the behavior of the economy in the ZLB interpreting the price frictions in the EW analysis as stemming from adjustment costs as proposed by Rotemberg (1982). A prominent example in this literature is Braun, Boneva and Waki (2015), who study the accuracy of linear approximations to the model.[13] This model is interesting because it implies the same linearized equations

[13]The Braun, Boneva, and Waki (2015) paper was first written in 2012. As best as we can tell, it is the first paper to analyze the accuracy of the linearized EW model of the ZLB relative to the

Figure 4.3: Multiplier with Shock to Expectations



that EW study. In this section we highlight an important potential shortcoming of using Rotemberg adjustment costs when studying multiplicity and learnability issues.

With one exception, the Rotemberg model is identical to the Calvo model discussed above. The exception is that instead of (2.7) - (2.8) we assume that the monopolist who produces the j^{th} good has the following objective:

$$E_t \sum_{k=0}^{\infty} \beta^k \lambda_{t+k} [(1+\nu) \frac{P_{j,t+k}}{P_{t+k}} Y_{j,t+k} - s_{t+k} Y_{j,t+k} - \Phi_{t+k} \left(\frac{P_{j,t+k}}{P_{j,t+k-1}} - 1 \right)^2]. \tag{5.1}$$

The variable Φ_t denotes a potentially state dependent function that scales the firm's costs of adjusting prices. In the classic Rotemberg model,

$$\Phi_t = \phi \tag{5.2}$$

To accommodate growth, Christiano and Eichenbaum (2012) assume

$$\Phi_t = \frac{\phi}{2}(C_t + G_t). \tag{5.3}$$

In contrast, authors like Braun et. al. (2015) and Gust, Herbst, Lopez-Salido and Smith (2015), assume

$$\Phi_t = \frac{\phi}{2} Y_t. \tag{5.4}$$

As it turns out, existence and learnability of equilibria in the Rotemberg model depend on exactly which specification of Φ_t one adopts.

It is well known that an interior minimum-state-variable rational expectations

underlying nonlinear model.

equilibrium for all three versions of the Rotemberg model is a set of eight numbers:

$$\pi, C, R, h, \pi_\ell, C_\ell, R_\ell, h_\ell,$$

that, when $r_t = r_\ell$, satisfy:

$$\begin{aligned}
R_\ell &= \max\left\{1, \frac{1}{\beta} + \alpha\left(\pi_\ell - 1\right)\right\} && (5.5)\\
\frac{1}{R_\ell} &= \frac{1}{1+r_\ell}\left[p\frac{C_\ell}{\pi_\ell C_\ell} + (1-p)\frac{C_\ell}{\pi C}\right] && (5.6)\\
h_\ell &= C_\ell + G_\ell + \Phi_\ell\left(\pi_\ell - 1\right)^2 && (5.7)\\
\left(\pi_\ell - 1\right)\pi_\ell &= \frac{1}{2\Phi_\ell}\varepsilon\left(\chi h_\ell C_\ell - 1\right)\left[C_\ell + G_\ell + \Phi_\ell\left(\pi_\ell - 1\right)^2\right] && (5.8)\\
&\quad + \frac{1}{1+r_\ell}\left[p\left(\pi_\ell - 1\right)\pi_\ell + (1-p)\left(\pi - 1\right)\pi\frac{C_\ell}{C}\frac{\Phi}{\Phi_\ell}\right]
\end{aligned}$$

Subscript ℓ denotes the value of a variable when $r_t = r_\ell$ and no subscript denotes the value of a variable after $r_t = r$.[14]

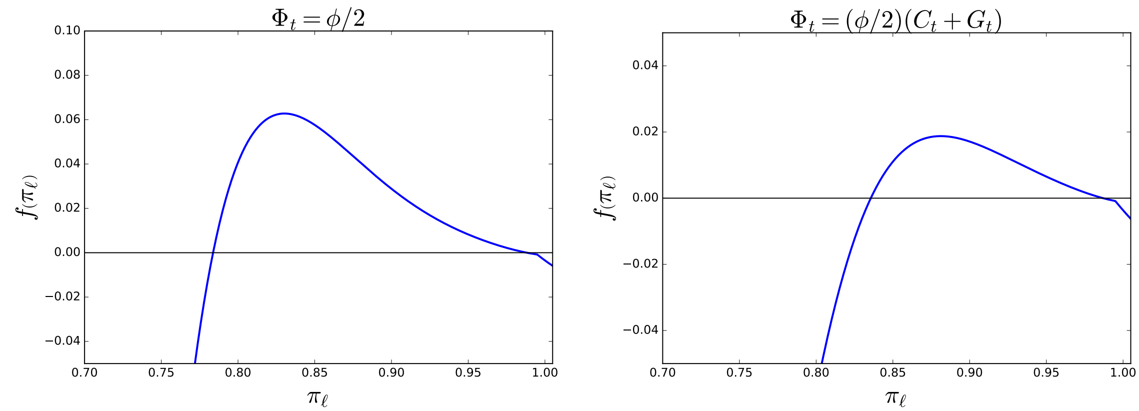
The equations defining a rational expectations equilibrium collapse into one equation in one unknown, π_ℓ,

$$f(\pi_\ell) = 0. \tag{5.9}$$

This equation is analogous to (2.15) in the Calvo model. The key difference is that the latter is an equation that determines candidate values for a ZLB steady state. Since there is no state variable in the Rotemberg model, conditional on prices and quantities after r_t once again is equal to r, equation (5.9) determines the equilibrium prices and quantities values so long as $r_t = r_\ell$.

[14]We formally derive these equations in the appendix and describe the way we solve for an equilibrium.

Figure 5.1: $f(\pi_\ell)$ in Rotemberg Model

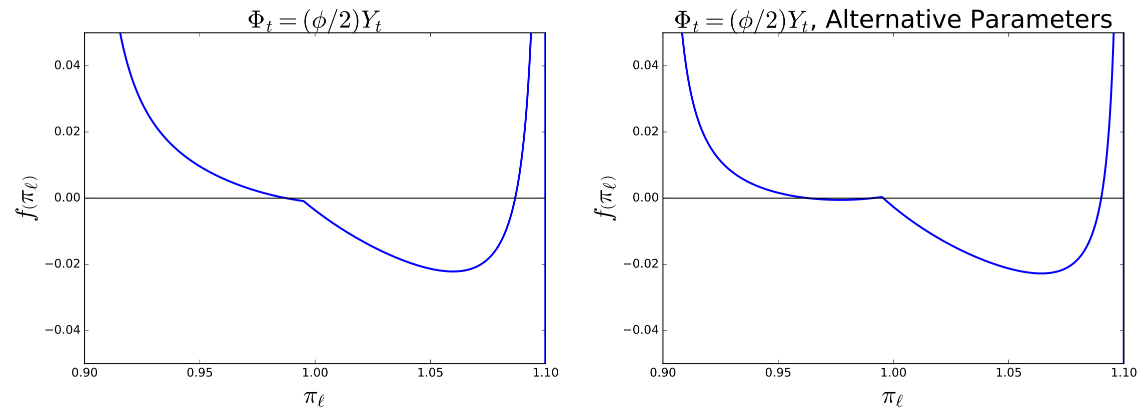


The two panels of Figure 5.1 plot $f(\pi_\ell)$ for Φ_t given by (5.2) and (5.3). In all cases we use the benchmark parameters given in (2.16). The parameter ϕ is chosen so that the log-linearized model implies the same system of equations implied by the log-linearized Calvo model, respectively.[15] The domain of admissible values of π_ℓ is restricted by the conditions that $C_\ell > 0$ and $Y_\ell > 0$. Two features of the figures are worth noting. First, the plots of $f(\pi_\ell)$ are very similar when Φ_t is given by (5.2) or (5.3). Second, there are two rational expectations equilibria in which the ZLB binds, both of which feature deflation. Note that the curve looks very similar to the analogous curve that determines the two ZLB steady state in the non-linear Calvo model.

Figure (5.2) displays $f(\pi_\ell)$ for the case where Φ_t is given by (5.4) and our benchmark parameter values. Notice that there are two equilibria when $r_t = r_\ell$. In one case the ZLB binds and that equilibrium is stable under learning. In the other case the ZLB doesn't bind. As it turns out with this specification of Φ_t it is possible to

[15] Given our normalization that steady state output is one, this requirement implies that ϕ satisfies $(\varepsilon - 1)\,\phi = \frac{(1-\theta)(1-\beta\theta)}{\theta}$.

Figure 5.2: $f(\pi_\ell)$ in Rotemberg Model



generate more exotic equilibria. The second panel of Figure 5.2 is the analog to the first except that model's parameters are given by:

$$\begin{aligned} \varepsilon &= 7.0,\ \beta = 0.99,\ \alpha = 2.0,\ p = 0.83, \\ r^\ell &= -0.0001,\ \phi = 200,\ \eta_g = 0.2, g_\ell = 0.23. \end{aligned}$$

Strikingly when $r_t = r_\ell$ there are now two equilibria where the ZLB binds and two equilibria where the ZLB isn't binding. This example is consistent with results in Braun et. al. (2015). Note that in both panels of Figure 5.2 $f(\pi_\ell)$ has asymptotes, at quarterly rates of deflation and inflation of 10%. At these rates of inflation, the costs of adjustment consume all of the output so that consumption can no longer be non-negative.

It is easy to characterize which ZLB equilibria are stable under learning for the Rotemberg model. Going from left to right in the plots, whenever $f(\pi_\ell)$ crosses from above, the equilibrium is stable under learning. From Figure 5.1, we see that when Φ_t is given by (5.2) or (5.3) there is a unique equilibrium that is stable under learning.

Table 5.1: Multipliers in the Rotemberg Model

Adj. Cost	Stable Equilibrium	Unstable Equilibrium
$\Phi_t = \frac{\phi}{2}$	1.56	0.98
$\Phi_t = \frac{\phi}{2}(C_t + G_t)$	1.70	0.36
$\Phi_t = \frac{\phi}{2}Y_t$	1.65	1.07

That equilibrium is the one with less deflation. When Φ_t is given by (5.4) and we work with the benchmark parameter values there is only one ZLB equilibrium and it is stable under learning. Notably, the non-ZLB equilibrium is not stable when adjustment costs are given by (5.4). Even with two ZLB equilibria, as in the second panel of 5.2, there is only one that is stable under learning. Interestingly, that equilibrium is the one that has more deflation. So our key conclusion from Calvo for the Rotemberg model: there is a unique rational expectations equilibrium at the ZLB that is stable under learning.

Unlike the Calvo model, the multiplier in the ZLB for the Rotemberg model is constant. Table 5.1 summarizes the values of the multiplier for the equilibria in Figures 5.1 and 5.2 that are stable. Notice that these multipliers are remarkably similar to each other and to the multiplier in the linear Calvo model (1.63). Viewed as a whole our results strongly support the view that once we focus on equilibria that are stable under learning, the implications of the NK model for multipliers in the ZLB are robust: the multiplier is large and increasing the more binding is the ZLB.[16]

[16] When adjustment costs are scaled by (5.2) or (5.3), we are able to find sunspot equilibria similar to the equilibria studied by Mertens and Ravn (2015). However, when adjustment costs are scaled by (5.4), there is no such ZLB equilibrium under our benchmark parameterization. Instead, the sunspot equilibrium exhibits high inflation. Again, we find that the sunspot equilibrium is not stable under learning.

6. Conclusion

In this paper we analyze whether the non-uniqueness of equilibria in NK models poses a substantive challenge to the key conclusions in the literature about the efficacy of fiscal policy in ZLB episodes. We argue that it does not. This conclusion rests on our view that if a rational expectations equilibrium is not stable-under learning, then it is simply too fragile to be a description of the data. We make our argument using particular models of learning. While we have explored alternative learning mechanisms, it is certainly possible that there exist alternative learning models for which our results do not go through. Still we believe our results are very supportive of the view that the key properties of linearized NK models regarding the impact of changes in government consumption in the ZLB are robust and should be taken seriously.

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