Intangibles, Inequality and Stagnation

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July, 2018

Abstract

We examine how aggregate output and income distribution interact with accumulation of intangible capital over time and across individuals. We consider an overlapping generations economy in which managerial skill (intangible capital) is essential for production, and it is acquired by young workers through on-the-job training by old managers. We show that, when young trainees are not committed to staying in the same firms and repaying their debt, a small difference in initial endowment and ability of young workers leads to a large inequality in accumulation of intangibles and income. Furthermore, a negative shock to endowment or the degree of commitment generates a persistent stagnation and a rise in inequality.

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1 Introduction

In the last few decades, especially after the global financial crisis of 2007-9, we observe two major concerns: slower growth of many countries and rising inequality across households within country. In Japan, there are heated debates on why Japan stopped growing and what caused the rising inequality after it entered into a prolonged financial crisis with the collapse of asset prices in the early 1990s. Although proposed explanations differ across researchers, the key phenomena to explain appear to be declining growth rate of total factor productivity and worsening labor market conditions for young workers.

In this paper, we explore a hypothesis that the slower productivity growth and the worsening youth labor market are entwined with intangible capital accumulation. For this purpose, we consider an overlapping generations economy in which managerial skill (intangible capital) is essential for production along with labor. Unlike physical capital, intangible capital - particularly managerial skill - cannot be directly transferred between generations. Young workers accumulate intangibles through on-the-job training offered by old managers. We formulate the technology of accumulating intangibles in a fairly general way: Inputs are the skill of old managers, the learning ability of young trainees and the amount of time both managers and trainees allocate for training; the outcome, managerial skill acquired by the young trainees, is subject to the idiosyncratic shocks.

Intangible capital also tends to be hard to be pledged as collateral. In our economy, managers offer young workers two options, a simple labor contract, which pays competitive wage without training, and a career path, which offers a compensation package and training to be future managers. The initial skill and wealth endowment are heterogeneous across young workers and are publicly observable. Idiosyncratic shocks to the outcome of intangible capital investment is also publicly observable. The career path thus can be conditioned on these information. If a trainee could commit to stay in the same firm and repay her debt, she would choose the option with a higher permanent income. If she chooses the career path to become a future manager, her consumption would be fully insured against the idiosyncratic shock to the outcome. Then, the training would only depend upon the initial skill and there would be no inequality in permanent and realized income, controlling for the initial skill.

In our baseline economy, however, the trainee is not committed to staying in the same firm. If she moves to another firm or starting a new firm, she will lose only a fraction of her managerial skill. The limited commitment affects the intangible capital investment and income distribution. Aggregate intangible investment is lower than that in the unconstrained economy for any given interest rate. Moreover, inequality in initial endowment of the young leads to diverse career paths and an unequal income distribution even among those with the same initial skill. At the extensive margin, rich young agents with large initial endowment opt for the career path to become future managers, while poor young workers receive no training and work as routine workers for life. At the intensive margin, richer young agents receive more intensive training to acquire better managerial skill, which leads to a large inequality even among workers who receive training. Insurance against idiosyncratic shocks to intangible capital accumulation is limited to downside risks, more more limited for poor young trainees. This incomplete insurance leads to a large inequality in realized income among managers with a long upward tail. Over time, a temporary adverse shock to initial endowment or the degree of commitment generates a persistent fall in intangible capital investment, aggregate production and rise in inequality.

The limited commitment is more severe when intangible capital becomes less firm-specific and moving across firms becomes easier for managers. This points to perhaps unintended consequences of liberalization of the labor market for skilled workers. Since European Union came into full force around 2000, skilled workers became more mobile across countries, especially from countries like Italy and Spain to countries like Germany and Britain. Before the 1990s financial crisis in Japan, Japanese skilled workers typically worked for the same firms for long time. This labor market condition changed after the crisis. Skilled workers switch jobs more often. While liberalization of the labor market of skilled workers improves match quality between workers and employers, the induced limited commitment may reduce skill acquisition on the job. In Japan, the fraction of young workers who got career-type permanent jobs declined relative to temporary jobs and career-type workers appear to receive less intensive on-the-job training after the crisis.[1]

Taking as given the limited commitment, our theory also provides some guidance for public policy. The competitive economy under limited commitment exhibits misallocation in matching between old managers and young workers with heterogeneous initial endowment and skill. Rich young workers receive more training regardless of their talent while poor but talented young workers receive less training under financing constraint. If the government

[1] Up to the early 1990s, Japanese large firms often sent their most promising career employees to the oversea graduate programs at the firms' expense. This practices largely disappeared since the late 1990s.

is better than private lenders in enforcing debt repayment so that it can relax the financing constraint, then it can provide loans for workers to receive training, which improves the resource allocation. If government is no better than private lenders in enforcing debtors (old managers) to pay, the policy option becomes more delicate. Government can provide subsidy for training poor young. But because government has difficulty in enforcing old managers to pay their liabilities (including tax liability), the subsidy must be financed by taxing workers (like payroll tax). Then the training subsidy may lead to too much training compared to the efficient allocation, which must be offset by the rationing of training based on the initial skill of young workers.[2]

Our paper is related to a few lines of literature. First, our model is based on Prescott and Boyd (1987) about firms as dynamic coalitions for intangible capital accumulation. Chari and Hopenhayn (1991) apply Prescott and Boyd (1987) for endogenous technology adoption, while Kim (2006) introduces financing constraint to Chari and Hopenhayn (1991) to show how difference in financing constraint leads to a large gap in TFP across countries. We introduce limited commitment and heterogeneous initial endowment and skill of young workers to Prescott and Boyd (1987). With these additional ingredients, we can study how small difference in initial conditions leads to a large inequality across workers and how a small shock to endowment or the degree of commitment leads to a persistent decrease in intangible capital accumulation and aggregate production.

Secondly related is a vast literature on wealth distribution, human capital accumulation and occupational choices in the presence of financial frictions. If we restrict attention to a most closely related literature, Galor and Zeira (1993) examine how indivisible human capital accumulation and financial friction lead to endogenous wealth distribution when parents care about their children and leave bequest. Banerjee and Newman (1993) show rich dynamics of wealth distribution and growth as a result of occupational choices. Although we have similar extensive margin of human capital accumulation through occupational choices, we introduce a richer technology for accumulating intangible capital which uses skills and time of managers and trainees as inputs for accumulating intangible capital. This leads to a richer

[2] If people can change the initial skill level at the start of working life through education, then people would start investing earlier to acquire better initial skill. Young people with larger initial endowment would have an advantage of acquiring initial skill through better education. Government can improve basic education to improve the initial skill, to create equal opportunity instead of equal outcome across all workers. This is related to Benabou (2002).

distribution dynamics through the matching between skilled managers and heterogeneous young workers.[3]

The third related literature is the macro literature on financial friction and capital misallocation. Kiyotaki (1998), Buera (2009), Buera, Kaboski, and Shin (2011) and Buera and Shin (2013) and Moll (2014) for example study how financial frictions affect misallocation of capital and economic growth. Our research is complementary to theirs because they focus on the allocation and accumulation of tangible capital and we focus on intangible capital. This addition is relevant because financial frictions may be more severe for intangible capital which is a large component of skilled workers' asset.[4]

Our theory is consistent with empirical findings on the level and the slope of workers' income profile in recent papers. Kambourov and Manovskii (2008) find that an increase in occupational mobility explains substantially why life-cycle earning profile becomes flatter, the experience premium becomes smaller and the inequality rises within group for more recent cohorts. While they emphasize the role of increasing occupation specific risks, we attribute the flattening life-cycle earning profile to the slowdown in investment in intangibles.[5] Guvenen et al. (2015) find that there is a strong positive association between the level of lifetime earning and how much earning grow over the life cycle.[6][7]

[3] On the other hand, we abstract from the endogenous bequest. See Banerjee and Duflo (2005)Banerjee and Duflo 2005 and Matsuyama (2007)Matsuyama 2007 for survey of more literature. See also Lucas (1992)Lucas 1992 and Ljungqvist and Sargent (2012)Ljungqvist and Sargent 2012 for a literature of endogenous financing constraints due to private information and hidden action, which we abstract in our model.

[4] Caggese and Perez-Orive (2017) study the implication of difference in collateralizability between intangible and tangible capital for the misallocation across firms. See also Eisfeldt and Papanikolaou (2013) for the asset price implications of organization capital - a specific form of intangible capital.

[5] Consistent with the theory, Heckman, Lochner, and Taber (1998) show that to account for skill premium, it is important to differentiate the potential income and the actual income during on-the-job training.

[6] Guiso, Pistaferri, and Schivardi (2013) find that firms operating in less financially developed markets offer lower entry wages but faster wage growth than firms in more financially developed markets, which is consistent with Michelacci and Quadrini (2009) in the earlier footnote. Guiso, Pistaferri, and Schivardi (2013) also find managers' income profile is steeper in financially underdeveloped market, which is consistent with our theory.

[7] Our framework is also motivated by literature on growth accounting, such as Corrado, Hulten, and Sichel (2009), which shows that intangible capital accumulation has become a dominant source of growth in labor productivity.

2 Basic Model

2.1 Framework

We develop an overlapping generations model of a production economy. Time is discrete and lasts forever. In each period, a unit measure of agents is born and lives for two periods. The expected utility of an agent born at date t is given by

$$V_t = \ln c_t^y + \beta \mathbb{E}_t(\ln c_{t+1}^o),$$

where c_t^y and c_{t+1}^o are consumption of homogeneous goods when young at date t and when old at date $t+1$. The parameter $\beta \in (0,1)$ is a utility discount factor, and $\mathbb{E}_t(\cdot)$ is the expected value of $\cdot$ conditional on date t information.

Young agents are heterogeneous in the initial endowment of learning ability κ and goods e.[8] These endowments are publicly observable, and follow a joint distribution, $F_t(\kappa, e)$ on $(\kappa, e) \in [0, \bar{\kappa}] \times [0, \bar{e}]$ at date t. Each agent is also endowed with a unit of time in each period and can work as a worker or a manager.

A firm is a dynamic coalition of current and future managers. There is a continuum of firms in the economy. Current managers of a firm jointly allocate their total managerial skill, which we call "intangible capital", to produce final goods and to train young agents to become future managers with intangible capital. If they allocate K^w units of intangibles and hire L units of labor, they can produce

$$y = A_t \left(K^w\right)^{\alpha} L^{1-\alpha} \tag{1}$$

final goods, where $\alpha \in (0,1)$ is the share of intangible capital in production and $A_t > 0$ is common aggregate productivity.

If current managers allocate $\tilde{k}$ units of intangible capital to train a young agent with learning ability κ and the trainee allocates $h \in [0,1]$ units of time for training, the trainee can acquire on average

$$k^+ = \frac{1}{b}\tilde{k}^{\eta} \left(h\kappa\right)^{1-\eta} \tag{2}$$

units of intangible capital in the next period. The parameter $\eta \in (0,1)$ is the share of current managers' input and $1-\eta$ is the share of trainee's input for acquiring intangible, and $b > 0$ is

[8] We consider e as exogenous, even though we could consider it as an endogenous inheritance from parents in an extension.

a common parameter for training cost.[9] The realization of intangible capital of the trainee, denoted k_z^+, is subject to an idiosyncratic shock, denoted z;

$$k_z^+ = zk^+, \tag{3}$$

where the shock z follows distribution function $\Phi(z)$, which has full support on $(0, \infty)$ and expected value 1. The intangible capital of old managers depreciates completely when they die.

The intangible capital of a manager is partially specific to the firm where she receives training: If she moves to another firm or starts a new firm in the next period, her intangibles shrink from k_z^+ to $(1-\theta)\, k_z^+$. The parameter $\theta \in (0,1)$ represents the specificity of intangibles. In the Basic Model, there is no shock to the aggregate economy nor firms, aside from an unanticipated shock, and there is no shock to the match quality between trainees and firms. In the Full Model of the next section, we introduce aggregate shocks and match quality shocks.

Young agents who do not receive training become routine workers in both periods of their life. Young agents who receive training will become future managers and cannot be routine workers when old. There is a competitive labor market, where routine workers supply labor to firms at wage rate, denoted w_t. There is also a competitive financial market, where risk free bonds that pay one unit of final good in the following period per unit are traded and priced at q_t.

Current managers of a representative firm decide final goods production, training and the compensation package for trainees. When the firm, with total intangible capital K_t from current managers, hires $n_t(\kappa, e)$ measure of young agents of ability and wealth endowment (κ, e), allocates $\tilde{k}_t(\kappa, e)$ intangibles to train each of them, and allocates K_t^w for final goods production, it must satisfy the capacity constraint as

$$K_t = K_t^w + \int \widetilde{k}_t(\kappa, e) n_t(\kappa, e) dF_t. \tag{4}$$

[9]This formulation follows Rothschild and White (1995)Rothschild and White 1995 on education. We consider the effort times the ability of the trainee, $h\kappa$, as an input and her expected intangible capital next period, k^+, as the output, the other input being the intangible capital of current managers (teachers). Different from Rothschild and White (1995), we ignore peer group effects among trainees and the training function shows constant return to scale. Since both trainees and managers spend time for intangible capital production, intangible capital production is also in the spirit of the Ben-Porath model of human capital accumulation. [ADD REFERENCE]

We can also think that the managers allocate K_t^w/K_t fraction of time for final goods production and the remaining fraction for training. When the trainee allocates time $h_t(\kappa,e)$ for training, it can acquire the intangibles according to $(2,3)$ as

$$k_{z,t+1}^{+}(\kappa,e)=z\frac{1}{b}\left[\tilde{k}_t(\kappa,e)\right]^{\eta}\left[h_t(\kappa,e)\kappa\right]^{1-\eta}. \tag{5}$$

When the firm hires L_t^w routine workers, the labor input equals the sum of the measure of routine workers and labor input from trainees - time allocated for production instead of training - as

$$L_t=L_t^w+\int\left[1-h_t(\kappa,e)\right]n_t(\kappa,e)dF_t. \tag{6}$$

The final goods output equals y in (1).

To recruit a trainee of type (κ,e) to be a future manager, the firm offers a compensation package consisting of consumption when young $c_t^y(\kappa,e)$, training to obtain intangible $k_{z,t+1}^{+}(\kappa,e)$ and consumption when old $c_{z,t+1}^{o}(\kappa,e)$, contingent on the realization of the idiosyncratic shock z. In order to recruit a trainee of type (κ,e), the compensation package has to be at least as good as her outside option

$$\ln c_t^y(\kappa,e)+\beta\int\ln c_{z,t+1}^{o}(\kappa,e)d\Phi(z)\geq V_t\left(\kappa,e\right),\forall(\kappa,e)\text{ such that }n_t(\kappa,e)>0, \tag{7}$$

where $V_t\left(\kappa,e\right)$ is the outside option (which we will specify shortly). Because firms can do everything the individual trainee can do to choose consumption through the financial market, we think of the firm as offering the compensation package which directly specifies the consumption plan of the trainee, $c_t^y(\kappa,e)$ and $c_{z,t+1}^{o}(\kappa,e)$.

The key friction in our economy is that, although the firm can commit to deliver the promised compensation, the trainee is free to walk away from the compensation package to work for another firm or start a new firm when old, losing a firm-specific fraction of her intangible capital.[10] Thus, the compensation package must satisfy the incentive constraint for the trainee to stay in the contract instead of leaving the firm:

$$c_{z,t+1}^{o}\left(\kappa,e\right)\geq(1-\theta)r_{t+1}k_{z,t+1}^{+}(\kappa,e),\forall z,\text{ and }(\kappa,e)\text{ such that }n_t(\kappa,e)>0, \tag{8}$$

where r_{t+1} is the rate of return on intangible capital if they work for other firms in the next period. As we will show shortly, the rate of return on intangible capital of all firms turns out

[10]This is one-sided limited commitment problem, with current managers committed to the contract and trainees facing ex post participation constraints, as in Kehoe and Levine (1993)Kehoe and Levine 1993 and Alvarez and Jermann (2000)Alvarez and Jermann 2000.

to be equal, because the final goods production function has constant return to scale and the labor market for routine workers is competitive (which equalizes the marginal product of intangible capital across final goods producers).

The present value of profit for the firm equals the sum of present gross profit (in the first line) and the net receipts from recruiting young agents for future managers (in the second line) as:

$$\begin{aligned} & A_t (K_t^w)^\alpha {L_t}^{1-\alpha} - w_t L_t^w \\ & + \int \left\{ -[c_t^y(\kappa, e) - e] + q_t \int \left[r_{t+1} k_{z,t+1}^+(\kappa, e) - c_{z,t+1}^o(\kappa, e) \right] d\Phi \right\} n_t(\kappa, e) dF_t. \end{aligned} \tag{9}$$

In the second line, the term $[c_t^y(\kappa, e) - e]$ is the compensation of the trainee when young, and the term $[r_{t+1} k_{z,t+1}^+(\kappa, e) - c_{z,t+1}^o(\kappa, e)]$ is the return on intangible capital minus compensation to the manager in future. The firm chooses final goods production K_t^w, L_t, L_t^w and the recruiting strategy of future managers $\left\{ n_t, c_t^y, h_t, \tilde{k}_t, k_{z,t+1}^+, c_{z,t+1}^o \right\} (\kappa, e)$ to maximize the present value of profit (9), subject to the constraints of intangible capital (4), intangible capital accumulation (5), labor (6), participation (7) and incentive (8).

Young routine workers can borrow from the financial market against their future labor income. The lifetime value of a routine worker of type (κ, e), $V_t^w(\kappa, e)$, is the solution to a standard problem of consumption and saving:

$$V_t^w(\kappa, e) = \max_{c_t^y, c_{t+1}^o \geq 0} \left[\ln c_t^y + \beta \ln c_{t+1}^o \right],$$

$$\text{subject to } c_t^y + q_t c_{t+1}^o = e + w_t + q_t w_{t+1}.$$

The outside option of the trainee of type (κ, e) is given by the maximum between $V_t^w(\kappa, e)$ and the highest expected utility offered by various firms as

$$V_t(\kappa, e) = \max \left\{ V_t^w(\kappa, e), \max_{all\ firms} \left[\ln c_t^y(\kappa, e) + \beta \int \ln c_{z,t+1}^o(\kappa, e) d\Phi(z) \right] \right\}. \tag{10}$$

The exogenous aggregate state is summarized by

$$s_t = (A_t, \theta_t, \omega_t),$$

where ω_t is a parameter which determines the distribution of initial endowment of young agents $F_t(\kappa, e) = F(\kappa, e \mid \omega_t)$. It turns out the endogenous aggregate state is summarized by the aggregate intangible capital stock K_t and measure of old routine workers (who did not receive the training when young) L_t^o.

Definition 1. A perfect foresight equilibrium is firms' policies K_t^w, L_t^w, L_t, $\left\{n_t, c_t^y, h_t, \widetilde{k}_t, k_{z,t+1}^+,\ c_{z,t+1}^o, V_t\right\}(\kappa$
routine worker's consumption plan $\left\{c_t^y, c_{z,t+1}^o\right\}(\kappa, e)$, wage rate w_t, the rate of return on intangibles r_t, bond price q_t as functions of aggregate state $S_t = (K_t, L_t^o, s_t)$ such that

a) Given prices (r_t, q_t, w_t) and the outside option $V_t(\kappa, e)$, firms' policy functions solve their problem;

b) Given firms' policies, outside option of the trainee is consistent with equilibrium (10), and labor and financial markets clear,

$$L_t^w = L_t^o + L_{t+1}^o, \tag{11}$$

$$\begin{aligned} \overline{e}_t + w_t &= \int_{\Theta_t} \left[w_t h_t(\kappa, e) + r_t \widetilde{k}_t(\kappa, e)\right] dF_t + \int c_t^y(\kappa, e) dF_t, \\ \Theta_t &= \{(\kappa, e) : n_t(\kappa, e) > 0\}; \end{aligned} \tag{12}$$

where $\overline{e}_t = \int e dF_t$.

c) K_{t+1} and L_{t+1}^o follow the laws of motion

$$L_{t+1}^o = \int [1 - n_t(\kappa, e)] dF_t(\kappa, e), \tag{13}$$

$$K_{t+1} = \int_{\Theta_t} \int k_{z,t+1}^+(e, \kappa) n_t(\kappa, e) d\Phi(z) dF_t(\kappa, e). \tag{14}$$

Because of the overlapping generations framework, the sum of endowment and wage in the left hand side (LHS) of (12) equals the sum of investment cost and consumption of young agents in the right hand side (RHS).

2.2 Intangible Capital Accumulation

To characterize the firm's policy function, let $\widetilde{r}_t$, $\lambda_t(\kappa, e)$ and $\mu_{z,t}(\kappa, e)$ be the Lagrangian multipliers of constraints on intangible capital (4) and participation (7) and incentive (8).

Solving (6) for L_t^w, the Lagrangian becomes

$$
\begin{aligned}
\mathcal{L} = {} & A_t(K_t^w)^\alpha {L_t}^{1-\alpha} - w_t\left[L_t - \int (1-h_t)n_t dF_t\right] \\
& + \int \left[e - c_t^y + q_t \int (r_{t+1}k_{z,t+1}^+ - c_{z,t+1}^o)d\Phi\right] n_t dF_t + \widetilde{r}_t\left[K_t - K_t^w - \int \widetilde{k}_t n_t dF_t\right] \\
& + \int \lambda_t(\kappa,e)\left[\ln c_t^y(\kappa,e) + \beta\int \ln c_{z,t+1}^o(\kappa,e)d\Phi(z) - V_t(\kappa,e)\right] n_t(\kappa,e)\, dF_t \\
& + \int\int \mu_{z,t}(\kappa,e)\left[c_{z,t+1}^o(\kappa,e) - (1-\theta)r_{t+1}k_{z,t+1}^+(\kappa,e)\right] d\Phi(z) n_t(\kappa,e)\, dF_t, \qquad (15)
\end{aligned}
$$

where $k_{z,t+1}^+(\kappa,e)$ satisfies (5). From the first order condition with respect to labor L_t, we learn

$$w_t = (1-\alpha)A_t(K_t^w/L_t)^\alpha.$$

From the first order condition with respect to intangible capital for production K_t^w, we learn

$$\widetilde{r}_t = \alpha A_t(L_t/K_t^w)^{1-\alpha} = \alpha\,(A_t)^{\frac{1}{\alpha}}\left(\frac{1-\alpha}{w_t}\right)^{\frac{1-\alpha}{\alpha}} \equiv r_t.$$

This verifies our earlier conjecture that the marginal product and the return on intangible capital $\widetilde{r}_t$ are equal across firms.

Let us define the minimum cost of training a young agent of type (κ,e) to acquire the expected level intangible capital k^+ as

$$\varphi_t(k^+;\kappa) = \min_{h,\tilde{k}} \;\left[w_t h + r_t\tilde{k},\right]$$
$$\text{subject to } k^+ = (1/b)\tilde{k}^\eta(h\kappa)^{1-\eta} \text{ and } 0 \le h \le 1.$$

The cost of acquiring intangibles is the opportunity cost for trainees to allocate h units of time and for current managers to allocate $\tilde{k}$ units of intangibles. As is illustrated in Figure 1, the training cost function is proportional to k^+ when the optimal time allocation $0 < h < 1$. It is convex in k^+ when $h = 1$. (All the details and derivations are in Section A.1 of the Appendix.)

$$\varphi_t(k^+;\kappa) = \begin{cases} \left(\frac{r_t}{\eta}\right)^\eta\left(\frac{w_t}{(1-\eta)\kappa}\right)^{1-\eta} bk^+, & \text{for } k^+ < \left(\frac{\eta}{1-\eta}\frac{w_t}{r_t}\right)^\eta \frac{\kappa^{1-\eta}}{b} \equiv \overline{k}(\kappa), \\ w_t + r_t\left[bk^+/\left(\kappa^{1-\eta}\right)\right]^{\frac{1}{\eta}}, & \text{for } k^+ \ge \overline{k}(\kappa). \end{cases} \qquad (16)$$

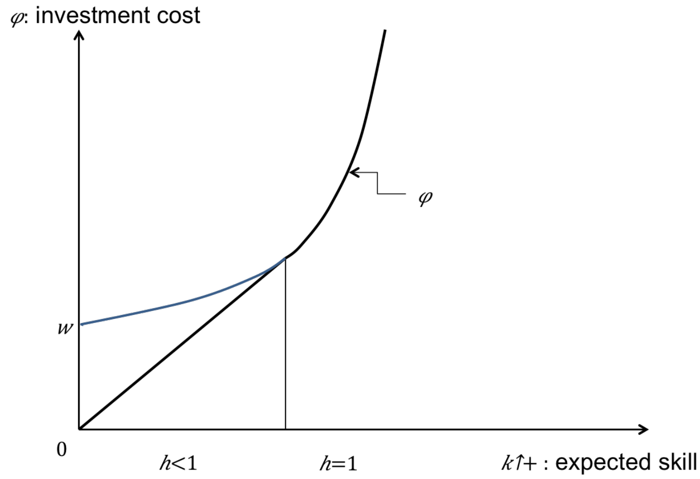

Figure 1: Training cost function $\varphi(k^{+};\kappa)$.

Let us define the net profit of recruiting a type-(κ, e) young for a future manager as

$$\begin{aligned}\pi_t(\kappa,e) &= e + w_t - c_t^y(\kappa,e) - \varphi_t(k_t^+(\kappa,e);\kappa) + q_t\int[r_{t+1}zk_t^+(\kappa,e) - c_{z,t+1}^o(\kappa,e)]d\Phi \\ &+ \lambda_t(\kappa,e)\left[\ln c_t^y(\kappa,e) + \beta\int \ln c_{z,t+1}^o(\kappa,e)d\Phi(z) - V_t(\kappa,e)\right] \\ &+ \int \mu_{z,t}(\kappa,e)\left[c_{z,t+1}^o(\kappa,e) - (1-\theta)r_{t+1}zk_t^+(\kappa,e)\right]d\Phi(z). \end{aligned} \tag{17}$$

The net profit function $\pi_t(\kappa, e)$ is the revenue minus the cost for recruiting a type (κ, e) young agent to be a future manager subject to her participation and incentive constraint constraints.

The Lagrangian of firm's policy function is now simply given by

$$\begin{aligned}\mathcal{L} &= r_tK_t + \left[A_t(K_t^w)^\alpha {L_t}^{1-\alpha} - w_tL_t - r_tK_t^w\right] \\ &+ \int \pi_t(\kappa,e)n_t(\kappa,e)dF_t. \end{aligned} \tag{18}$$

Because the competition among firms under constant returns to scale technology for production and training, the net profit function cannot be positive in equilibrium and

$$\begin{aligned}& n_t(\kappa,e) > 0 \text{ implies } \pi_t(\kappa,e) = 0, \text{ and} \qquad (19)\\ & \pi_t(\kappa,e) < 0 \text{ implies } n_t(\kappa,e) = 0.\end{aligned}$$

Also, the net profit from final goods production equals zero in the term in square bracket in the first line of the Lagrangian (18). Therefore the present value of profit equals the total returns on firm's intangible capital, $r_t K_t$.

The profit maximization problem has an equivalent dual problem which looks perhaps more familiar. The equivalent formulation is an optimal consumption-investment decision by a type-(κ, e) trainee.

$$\max_{c_t^y(\kappa,e),k^+(\kappa,e),c_z^o(\kappa,e)} \left[\ln c_t^y(\kappa, e) + \beta \int \ln c_{z,t+1}^o(\kappa, e) d\Phi(z)\right] \tag{20}$$

$$\text{s.t. } c_t^y(\kappa, e) + \varphi_t(k^{+(\kappa,e)}; \kappa) + q_t \int c_{z,t+1}^o (\kappa, e)\, d\Phi = e + w_t + q_t r_{t+1} k_t^+(\kappa, e), \tag{21}$$

$$c_{z,t}^o (\kappa, e) \geq (1 - \theta) r_{t+1} z k_t^+(\kappa, e), \forall z. \tag{22}$$

(21) is the trainee's lifetime budget constraint. Notice that, if the trainee is hired by a representative firm, the net profit must be zero from (19) , which implies (21) . Because the utility function of the trainee is concave, she has an incentive to smooth consumption across time and states. The constraint on commitment due to the trainee's freedom to walk away from the contract (22) limits her ability to borrow against future income and to insure against the idiosyncratic shock.

To understand how the limited commitment constraint affects agents' intangible capital accumulation, consider first a full-commitment benchmark, the problem without constraint (22). From the first order condition of the utility maximization (20) subject to the budget constraint (21) only, we learn

$$\varphi_t'(k_t^+ (\kappa, e)\, ; \kappa) = q_t r_{t+1}, \text{ for all trainees.} \tag{23}$$

Here, the marginal cost of intangible investment (for acquiring an additional unit of expected intangible) in the LHS equals the discounted expected rate of return on intangibles in the RHS. Because the marginal cost of intangible investment is strictly decreasing with learning ability κ, we learn that the trainee will devote all her time to learning, $h(\kappa, e) = 1$,[11] and that this marginal condition becomes

$$q_t r_{t+1} = \frac{b^{\frac{1}{\eta}}}{\eta} r_t \left(\frac{k_t^+ (\kappa, e)}{\kappa}\right)^{\frac{1-\eta}{\eta}}, \text{ or}$$

[11] As will be shown shortly, only young agents with learning ability higher than a threshold κ_t^* are trained. And all trainees, except for possibly $\kappa = \kappa_t^*$, choose $h_t = 1$ from (23), and the exception is of measure zero.

$$k_t^+(\kappa, e) = a_t^* \cdot \kappa, \text{ where } a_t^* \equiv \left(\frac{\eta q_t r_{t+1}}{b^{1/\eta} r_t}\right)^{\frac{\eta}{1-\eta}}$$

from (16). Thus, the expected value of intangibles for a trainee is proportional to her learning ability. Concerning the consumption, her consumption when old is independent of the idiosyncratic productivity shock because of insurance and it satisfies the Euler equation with her consumption when young:

$$c_{z,t+1}^o(\kappa, e) = c_{t+1}^o(\kappa, e) = \frac{\beta}{q_t} c_t^y(\kappa, e), \text{ for all } z.$$

The budget constraint (21) can be rewritten as

$$(1+\beta) c_t^y(\kappa, e) = e + (1-\eta) q_t r_{t+1} k_t^+(\kappa, e).$$

The LHS is the present value of consumption, while the RHS is the net worth, the sum of the initial endowment and the trainee's share $(1-\eta)$ of the expected present value of return from the intangible investment $q_t r_{t+1} k_t^+(\kappa, e)$. The remaining η share belongs to the current managers, reflecting the intangible capital accumulation function (2).

Concerning the extensive margin on who becomes a trainee rather than a routine worker, we can compare the net worth of the trainee with that of a routine worker $e + w_t + q_t w_{t+1}$. Thus a young agent of type-(κ, e) becomes a trainee if and only if

$$e + (1-\eta) q_t r_{t+1} k_t^+(\kappa, e) > e + w_t + q_t w_{t+1}, \text{ or}$$

$$\kappa > \kappa_t^* \equiv \frac{w_t + q_t w_{t+1}}{(1-\eta) q_t r_{t+1} a_t^*}.$$

The threshold κ_t^* is independent of the initial wealth, e.

To summarize, heterogeneity in initial wealth endowment and the idiosyncratic shock to intangible do not affect young worker's occupational choice and the intensity of intangible capital investment, and the idiosyncratic shock does not affect consumption under full commitment.

Under limited commitment, in contrast, the consumption of a trainee has to satisfy the incentive constraint (22), so that consumption will depend on the idiosyncratic productivity shock as

$$c_{z,t+1}^o = \begin{cases} (1-\theta) r_{t+1} z k_t^+(\kappa, e), & \text{for } z > z_t^*(\kappa, e), \\ (1-\theta) r_{t+1} z_t^*(\kappa, e) k_t^+(\kappa, e), & \text{for } z \leq z_t^*(\kappa, e), \end{cases} \tag{24}$$

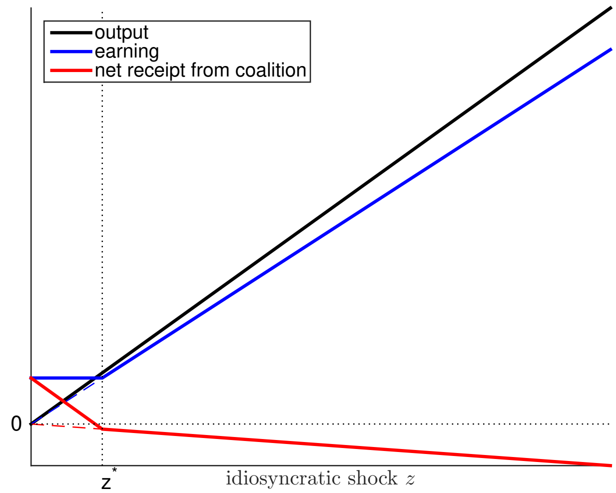

Figure 2: Manager's earnings and output across idiosyncratic shock state z.

where

$$(1-\theta)r_{t+1}z_t^*(\kappa,e)k_t^+(\kappa,e) = \frac{\beta}{q_t}c_t^y(\kappa,e). \tag{25}$$

Thus, her consumption when old is insured against the downside risk, when $z \leq z_t^*(\kappa,e)$, but not against the upside risk of the idiosyncratic shock $z > z_t^*(\kappa,e)$. Moreover, because the lowest consumption level when old is consistent with consumption when young under permanent income theory, the consumption-age profile tends to be upward-sloping, similar to that in financing constraint models. See Figure 2.

The first order condition for intangible investment is now

$$\varphi_t'(k_t^+(\kappa,e),\kappa) = r_{t+1}Q_t(\kappa,e), \text{ where} \tag{26}$$

$$Q_t(z_t^*(\kappa,e)) \equiv q_t\left\{1-(1-\theta)\int_{z_t^*(\kappa,e)}^{\infty}[z-z_t^*(\kappa,e)]d\Phi(z)\right\}. \tag{27}$$

We can think of $Q_t(z_t^*(\kappa,e))$ as the effective discount factor for intangible investment for type-(κ,e) trainee, taking into account the undiversifiable upside risk due to limited commitment, which is an increasing function of $z_t^*(\kappa,e)$. Comparing (26) with the first order condition under full commitment (23), we learn the discounted expected marginal return on intangibles is suppressed due to the limited commitment and that intangible investment is lower than that in the first best allocation for the same price level.

From $(21, 24, 26)$, we also learn that

$$(1+\beta)\, c^y(\kappa, e) + \varphi_t(k_t^+(\kappa, e), \kappa) = e + w_t + r_{t+1} Q_t\left(z_t^*(\kappa, e)\right) k_t^+(\kappa, e). \tag{28}$$

The compensation package of trainee of type (κ, e) (or trainee's choice) is summarized by $\left\{c_t^y, k_t^+, z_t^*, c_{z,t+1}^o\right\}(\kappa, e)$ which satisfy $(24, 25, 26, 28)$. The discounted expected utility of future managers of type-(κ, e) is

$$\begin{aligned}
V_t^m(\kappa, e) &= \ln c_t^y(\kappa, e) + \beta \int \ln c_{z,t+1}^o(\kappa, e) d\Phi \\
&= (1+\beta) \ln\left[e + w_t + r_{t+1} Q_t\left(z_t^*(\kappa, e)\right) k_t^+(\kappa, e) - \varphi_t(k_t^+(\kappa, e), \kappa)\right] \\
&\quad + \beta \int \left[\ln z - \ln z_t^*(\kappa, e)\right] d\Phi + \beta \ln\left(\frac{\beta}{q_t}\right) - (1+\beta)\ln(1+\beta).
\end{aligned}$$

When $k_t^+(\kappa, e)$ is small enough, the trainee splits her time between training and production. The marginal cost of intangible investment is constant from (16) so that (26) determines the threshold idiosyncratic shock $z_t^*(\kappa, e)$ below which the consumption is insured as

$$\left(\frac{r_t}{\eta}\right)^{\eta} \left(\frac{w_t/\kappa}{1-\eta}\right)^{1-\eta} b = r_{t+1} Q_t\left(z_t^*(\kappa, e)\right).$$

Because LHS is a decreasing function of learning ability of the trainee, κ, and the RHS is an increasing function of $z_t^*(\kappa, e)$, we learn that $z_t^*(\kappa, e)$ is a decreasing function of κ. Because the investment cost function is proportional to the scale in this case, we know $\varphi_t(k^+; \kappa) = \varphi_t'(k^+; \kappa) k^+$ and the budget constraint (28) implies

$$e + w_t = (1+\beta)\, c_t^y(\kappa, e) = \frac{1+\beta}{\beta}(1-\theta) q_t r_{t+1} z_t^*(\kappa, e) k_t^+(\kappa, e). \tag{29}$$

Intangible investment is now proportional to endowment plus wage.[12] Intangible investment also becomes an increasing function of the learning ability κ as $z_t^*(\kappa, e)$ is a decreasing function of κ :

$$\frac{\partial}{\partial \kappa} z_t^*(\kappa, e) < 0,\ \frac{\partial}{\partial \kappa} k_t^+(\kappa, e) > 0,\ \frac{\partial}{\partial e} k_t^+(\kappa, e) > 0.$$

Under the limited commitment, a more talented trainee sacrifices her consumption when young and consumption insurance when old in order to increase her intangible investment.

[12] When the trainee splits her time for training and working, she does not earn "profit" from intangibles investment because the training cost function is proportional to the intangible investment. Instead, her net worth becomes sum of value of endowment of goods and time, $e + w_t$, which is independent of her learning ability.

Wealthier trainee invests more in intangible capital, or self selects a compensation package or job which involves a larger intangible investment.

When initial endowment e is large enough relative to ability κ, the trainee spends all her time for training. Intangible investment exceeds the threshold $\overline{k_t}(\kappa)$ so that the investment cost function (16) becomes convex. In this case, the marginal condition (26) and the budget constraint (28) becomes

$$b^{\frac{1}{\eta}} \frac{r_t}{\eta} \left(\frac{k_t^+(\kappa, e)}{\kappa} \right)^{\frac{1-\eta}{\eta}} = \varphi'(k^+(\kappa, e); \kappa) = r_{t+1} Q_t \left(z_t^*(\kappa, e) \right), \text{ or}$$

$$k_t^+(\kappa, e) = a_t^{**}(z_t^*(\kappa, e)) \cdot \kappa, \text{ where}$$
$$a_t^{**}(z_t^*(\kappa, e)) \equiv \left[\frac{\eta r_{t+1} Q_t \left(z_t^*(\kappa, e) \right)}{b^{\frac{1}{\eta}} r_t} \right]^{\frac{\eta}{1-\eta}}.$$

$$e + (1-\eta)\varphi'(k^+(\kappa, e); \kappa) k^+(\kappa, e) = (1+\beta) c^y(\kappa, e) = \frac{1+\beta}{\beta} (1-\theta) q_t r_{t+1} z^*(\kappa, e) k_t^+(\kappa, e).$$

[13]From these we learn

$$\frac{\partial}{\partial \kappa} z^*(\kappa, e) < 0, \; \frac{\partial}{\partial \kappa} k_t^+(\kappa, e) > 0, \; \frac{\partial}{\partial e} z^*(\kappa, e) > 0, \; \frac{\partial}{\partial e} k_t^+(\kappa, e) > 0.$$

As before, a more talented trainee sacrifices smooth consumption across time and states to increase intangible investment, while a wealthier trainee invests more in intangible capital. In addition, a wealthier trainee consumes more when young and insures more when old. See Figure **??**.[14]

[13]When the trainee spends all her time for training, she receive "profit" from intangibles investment, which is an increasing function of her ability. Her net worth becomes sum of value of endowment of goods and the profit, similar to the budget constraint under full commitment.

[14]As is illustrated in 1, the allocation of managers' intangible capital is strictly increasing function of trainee's initial skill and endowment. This may remind the reader of assortative matching between trainee's initial skill, goods endowment and manager's productivity. The assortative matching result is similar to that in Anderson and Smith (2010). We relax an assumption in Anderson and Smith (2010), that matching is one-to-one. Instead, a trainee can rent intangible capital from multiple managers and a manager can train multiple trainees. This makes the model more tractable. The distribution of intangible capital across managers is not an aggregate state variable. The aggregate amount of intangible capital and routine workers are the only endogenous state variables.

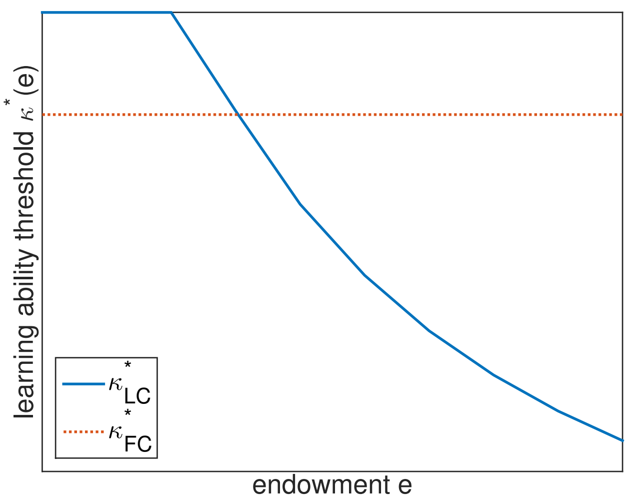

Figure 3: Occupational choice of young workers.

Concerning the occupational choice, type-(κ, e) agent chooses to become future manager if and only if

$$V_t^m(\kappa, e) > V_t^w(\kappa, e),$$

or

$$(1+\beta)\ln\left[e + w_t + r_{t+1}Q_t\left(z_t^*(\kappa, e)\right)k_t^+\left(\kappa, e\right) - \varphi_t(k_t^+\left(\kappa, e\right), \kappa)\right]$$
$$+ \beta\int_{z_t^*(\kappa,e)}^{\infty}\left[\ln z - \ln z_t^*(\kappa, e)\right]d\Phi > (1+\beta)\ln(e + w_t + q_t w_{t+1}).$$

Thus we learn that only talented and relatively wealthy young agents become trainee. As is illustrated in Figure 3, the threshold ability κ_{LC}^* for young agents to receive training is a decreasing function of endowment. Unlike in the economy under full commitment (in which the threshold is κ_{FC}^* does not depend upon wealth), poor young agents will not receive training to become future managers even if they are very talented.

2.3 Income Inequality

Because the initial endowment of ability and wealth affects the occupational choice at the extensive margin and intangibles investment at the intensive margin, and the outcome of

fraction of positive endowment ω	0.7
learning ability distribution $H(\kappa)$	$U[0,1]$
share of intangibles α	0.3
share parameter of manager's skill η	0.5
utility discount factor β	0.75
specificity of intangible capital θ	0.1
standard deviation of idiosyncratic shock	1

Table 1: Parameter values used in model simulation

intangibles investment leads to partially uninsurable idiosyncratic risk, we explore the implication of the Basic Model for inequality in the lifetime permanent income and realized income. In this section, we use numerical examples to illustrate the effect of limited commitment on income inequality. The parameter values in the numerical example are reported in Table 1. We assume that initial skill and goods endowment are independent from each other, $F_t(\kappa, e) = G_t(e)H(\kappa)$. There is a mass, $1 - \omega_t$, of agents with no initial goods endowment. Conditional on receiving positive wealth endowment, the endowment is uniformly distributed between 0 and $\bar{e}$:

$$G_t(e) = \begin{cases} 1 - \omega_t, & \text{for } e = 0, \\ 1 - \omega_t(\bar{e} - e)/\bar{e}, & \text{for } 0 < e \leq \bar{e}. \end{cases}$$

Most other parameters are relatively standard. Examples in later sections are also computed using these parameter values as a benchmark.

In the steady state, the expected present value of life-time income, which we call "permanent income," is given by:

$$\mathcal{Y}(\kappa, e) = \begin{cases} w(1+q), & \text{if } \kappa \leq \kappa^*(e), \\ c^y(\kappa, e) + q\mathbb{E}c_z^o(\kappa, e) - e, & \text{if } \kappa > \kappa^*(e). \end{cases}$$

Under full commitment, the threshold for the occupational choice $\kappa^*(e)$ is independent of e. Young agents' occupational choice depends only on their present value of income. An agent chooses to be trained if and only if the permanent income from being a trainee is higher than that from being a routine worker for life. Among trainees, their permanent income is an increasing function of their learning ability. These features are illustrated in Figure 4.

The permanent income of the most talented agent is about 15% higher than that of a routine worker in our numerical example. Intangible capital accumulation does not induce too much inequality in permanent income. The inequality in realized income is also modest because managers fully share the risk of idiosyncratic shocks to their intangible accumulation.

Under limited commitment, in contrast, the initial endowment of skill and wealth have a much bigger effect on their permanent income through their effect on intangible capital accumulation at the extensive and intensive margin. Trainees have an upward-sloping consumption profile and are exposed partially to risks of intangible capital outcome. To compensate for the non-smooth consumption across time and states, a premium in the permanent income of a trainee arises and it is increasing in the intensity of intangible capital accumulation. These features are illustrated in Figure 5. Young workers with initial skill and wealth endowment of $(\kappa^*(e), e)$ are indifferent between being a routine worker and a trainee. In order to make them indifferent, they need to receive premia in permanent income if they choose to be trained. Moreover, the premia in permanent income at the threshold is larger for those who have smaller wealth endowment, because they reduce consumption when young and consumption insurance when old to finance intangible investment more. In Figure 5, young with lower endowment needs a higher ability to become a trainee and has a larger vertical jump in the permanent income when she switch from a routine worker to a future manager.

Controlling for the initial skill, a trainee with higher initial wealth accumulates more intangible capital and receives a larger premium in compensation for riskier consumption profile. The limited commitment magnifies the effect of difference in initial endowment of skill and wealth on the permanent income: the most privileged young agent with largest endowment of skill and wealth enjoyed 180% larger permanent income relative to routine workers under limited commitment, while the largest gap is only 15% under full commitment.[15]

Inequality in realized income is even larger than the expected present value of income, because trainees are only partially insured against idiosyncratic shocks. In Figure 6, we illustrate the present value of the realized income range from the lowest 10 percentile to the highest 10 percentile as the vertical line with two short horizontal marks for young agents

[15] In our example of relatively small initial wealth e and specificity of intangibles θ, all trainees work partly for production $h_t < 1$.

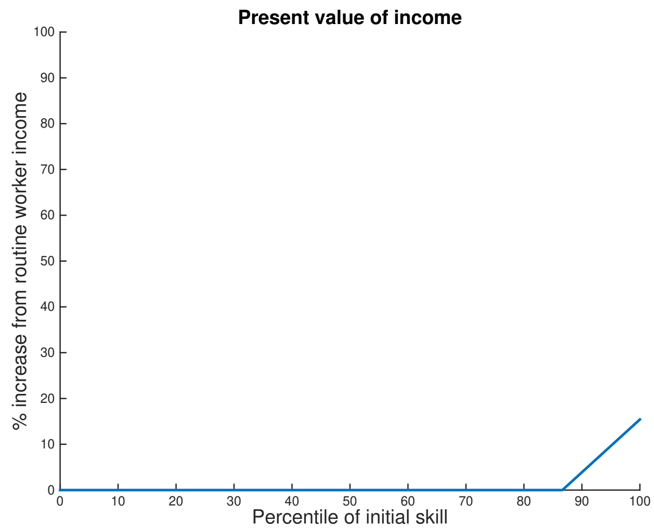

Figure 4: Distribution of permanent income under full commitment.

with different initial skill ($\kappa \in [0,1]$) for three levels of initial endowment ($e = 1$, 0.67, and 0.33). The solid lines are the permanent income for three initial endowment (extracted from the previous Figure). The lowest 10 percentile income realization is about 40% lower than the routine worker with same endowment, which does not depend upon the initial skill nor endowment with log utility function from (29). The top 5 percentile realization of a trainee with the highest learning ability and highest initial goods endowment is as high as 800% more than the permanent income of a routine worker. The gap in the present value of realized income for top 1 percentile is much more higher than 800%.[16]

2.4 Stagnation

In this section, we study how limited commitment propagates the effect of aggregate shocks to the economy over time, instead of propagating the effects of difference in the initial condition on the outcome across agents.

[16]We assume the idiosyncratic shock z has Gamma distribution with a long upside tail for numerical examples.

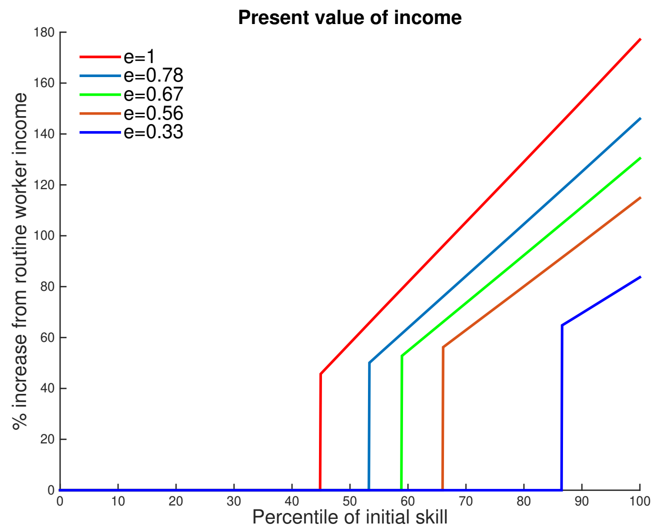

Figure 5: Distribution of permanent income under limited commitment.

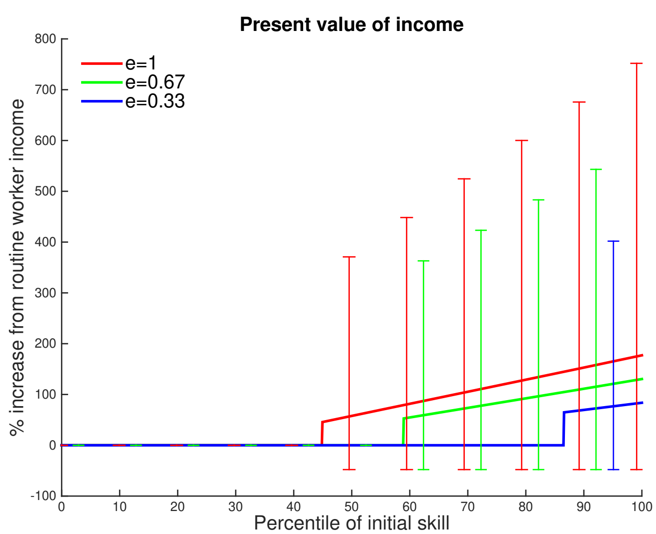

Figure 6: Distribution of realized income under limited commitment.

2.4.1 Shock to Endowment Distribution

As the first experiment, we examine the aggregate effects of an unexpected negative shock to agents' wealth endowment. This is meant to capture the effect of collapse of asset values and wealth endowment perhaps due to a financial crisis. The negative shock to endowment is modeled as a shock to the total measure of young agents with positive endowment, ω_t, keeping fixed the conditional distribution of young agents with positive endowment. Initially ω_t drops by 10% from 0.7 to 0.63. After the initial shock, ω_t converges gradually to the original level, with a half life of about 2 periods.

Figure 7 illustrates the dynamic responses of intangible capital, K_t, output, Y_t, return on intangible capital r_t, and the share of intangible capital used for training $(K_t - K_t^w)/K_t$. The dotted lines are aggregate responses in an unconstrained economy where there is no constraint on commitment. The recession in the constrained economy is deeper in magnitude. The drop in aggregate output is about 1% in the economy with limited commitment while it is about 0.5% in the unconstrained benchmark. The deeper recession is induced by tighter financing constraint that arises from limited commitment and reduced endowment.

With only the endowment shock, the recovery in the constrained economy is not slower than that in the unconstrained economy. This is because under this calibration, all trainees spend less than their whole working time for training. Adjusting hours of training minimizes the effect of misallocation.

2.4.2 Negative Shock to Specificity of the Intangible Capital

As the second experiment, we study the impact of an unexpected and permanent decrease in the intangible capital. This tries to illustrate the effect of changes in the labor market. During "the lost two decades" of early-1990s and early-2010s in Japan, their labor market underwent a structural change: the relationship between workers and firms becomes less likely to last for life-time, and permanent workers are more mobile with the development of labor market for mid-career workers - a sign of declining specificity of intangible capital. The negative shock to the specificity of intangible capital is modeled as a shock to θ_t, which reduces θ_t permanently by 10% from 0.1 to 0.09.

When the specificity decreases, aggregate intangible capital stock and output decrease significantly and persistently with limited commitment. In contrast, in an unconstrained benchmark, all equilibrium variables remains constant.

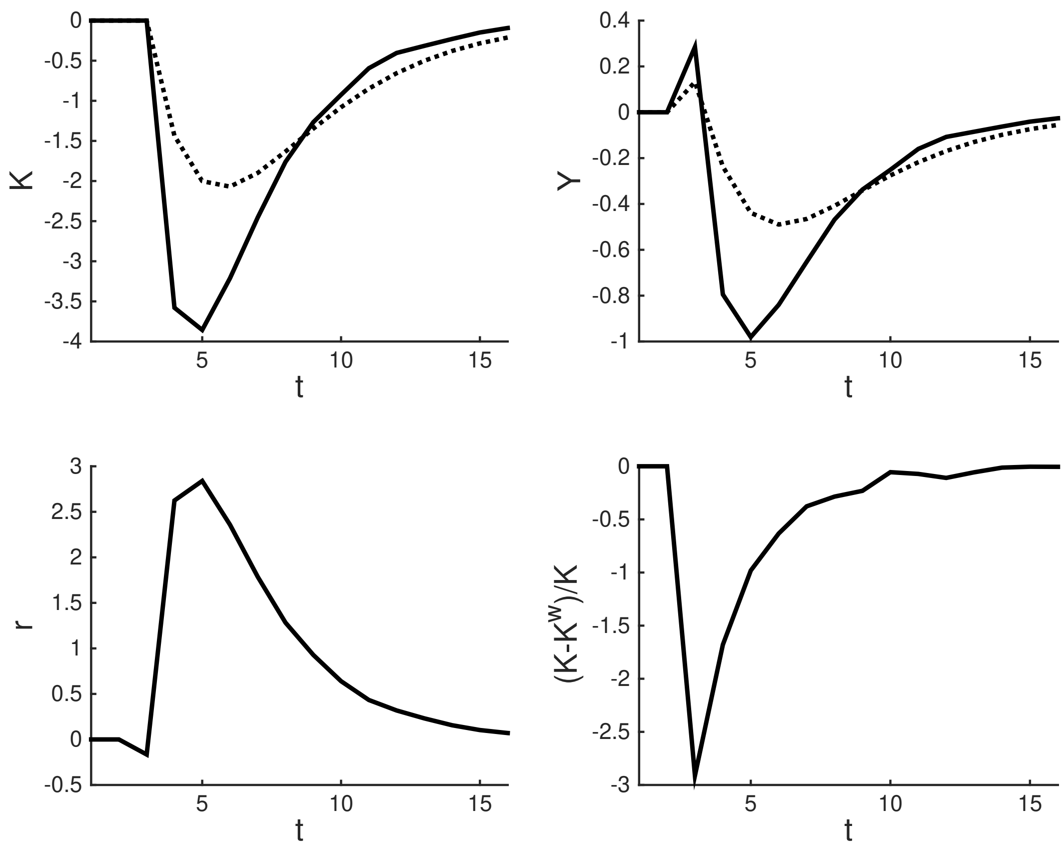

Figure 7: Dynamic response to negative endowment shock.

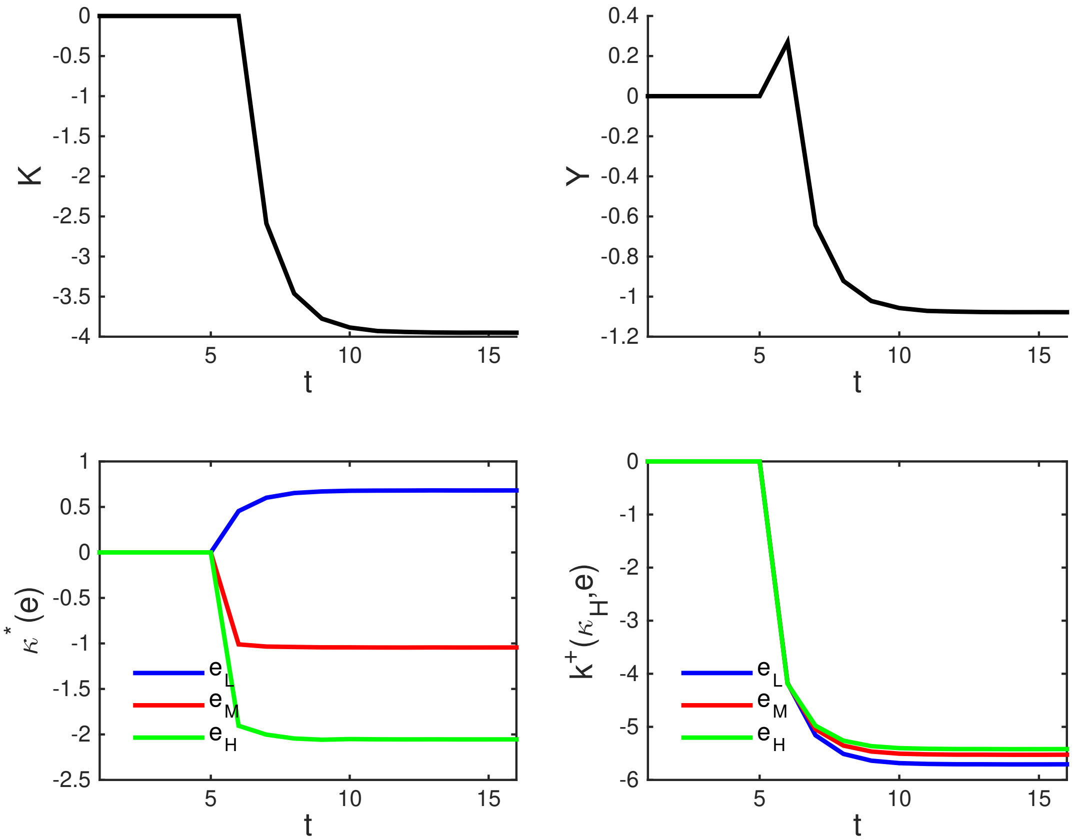

Figure 8: Experiment 2: Dynamic response to negative shock to commitment.

The misallocation of intangible capital on extensive and intensive margin along the transition path is clearer than in Experiment 1. $\kappa_t^*(e)$ drops by more than 2% for agents with high wealth endowment while $\kappa_t^*(e)$ increases by .7% for agents with low wealth endowment. Among agents with high skill, the decline in intangible capital accumulation on the intensive margin is more severe for agents with low wealth endowment. Over time, the intangible capital accumulation drops by 5.8% for those with low wealth endowment while it drops by 5.4% for those with high wealth endowment.

2.5 Efficiency

In this section, we examine whether the equilibrium allocation is constrained efficient in equilibrium. For purpose, we look for the Pareto weight for an agent of type (κ, e) born at period t, $\gamma_t(\kappa, e)$, with which the social planner's problem corresponds to the competitive equilibrium. The social planner's objective at period t is

$$\int \gamma_{t-1}(\kappa, e) \int \ln c^o_{zt}(\kappa, e) d\Phi dF + \sum_{\tau=t}^{\infty} \int \gamma_\tau(\kappa, e) \left[\ln c^y_\tau(\kappa, e) + \beta \int \ln c^o_{z\tau+1}(\kappa, e) d\Phi \right] dF$$

The planner faces constraints on intangible capital (4), labor (6), aggregate resource constraints,

$$\int \int c^o_{zt}(\kappa, e) d\Phi dF + \int c^y_t(\kappa, e) dF = \bar{e}_t + A_t \left(K^w_t\right)^\alpha \left(L^w_t\right)^{1-\alpha},$$

laws of motion of routine workers and intangibles, (13) and (14), and managers' incentive constraints, (8). In the Appendix, we show the competitive equilibrium achieves a constrained efficient allocation with suitable Pareto weights.

2.6 National Account and Labor Share

For the economy with significant intangible investment, we need to take into account the unique aspects of intangible investment for the System of National Account. Because our model is an overlapping generations model with two-period lifetime, it is not suitable for measuring annual or quarterly GDP. Here we only discuss the qualitative feature of how to measure GDP and labor share according to the Basic Model.

The System of National Account measures the aggregate economic activity from expenditure, production and distribution. From the expenditure side, abstracting from the government and the foreign sector, we measure gross domestic expenditure as the sum of consumption and investment as:

$$GDE_t = \int c^y_t (\kappa, e)\, dF_{t-1} + \int \int c^o_{z,t} (\kappa, e)\, d\Phi dF_t + \int r_{t+1} Q_t(\kappa, e) k^+_t(\kappa, e) dF_t,$$

where $Q_t(\kappa, e)$ is the effective discount factor of type-(κ, e) agent given by (27). Here, we measure gross investment as the sum of intangible investment by future managers, taking into account that intangible capital depreciates completely with death of old agents, and measure them in terms of value rather than cost of investment. From the production side, we can define the gross domestic product as the sum of value added. Including the value of initial goods endowment of young agents as output of home production, the gross domestic product is

$$GDP_t = A_t \left(K^w_t\right)^\alpha L^{1-\alpha}_t + \int r_{t+1} Q_t(\kappa, e) k^+_t(\kappa, e) dF_t + \int e dF_t,$$

From income distribution side, we define gross domestic income as the sum of wages, return on intangible capital, profit (rent) from intangible investment and home production as

$$GDI_t = w_t \left(L_t^o + 1\right) + r_t K_t + \int \left[r_{t+1} Q_t(\kappa, e) k_t^+(\kappa, e) - \varphi_t \left(k_t^+(\kappa, e); \kappa \right) \right] dF_t + \int e dF_t.$$

From market clearing condition, with $A_t \left(K_t^w\right)^\alpha L_t^{1-\alpha} = w_t L_t + r_t K_t^w$ and $\varphi_t \left(k_t^+(\kappa, e); \kappa \right) = w_t h_t(\kappa, e) + r_t \widetilde{k}_t(\kappa, e)$, we learn the usual equality of national income from production, expenditure and distribution as:

$$GDP_t = GDE_t = GDI_t.$$

Although the above System of National Account is consistent with our theoretical framework, it is difficult to disentangle managers' total compensation between "wage" and "return on intangible investment" in practice. Thus, it is often measure the gap between manager's total compensation and opportunity wage as "profit."

$$\text{measured "profit" of old manager} \;=\; c_{z,t}^o - w_t.$$

The measured "investment" of young trainee could be $e + w_t - c_t^y$. Under full commitment, such measurement may not be too misleading, and the present value of measured profit reflects the scarcity of young agents with higher learning ability, as shown in Figure 4.

In contrast, when the limited commitment influences the investment and returns on intangible investment, measuring "profit" as the gap between manager's total compensation minus opportunity wage is very misleading. The measured profit under limited commitment includes uninsurable realized returns on intangible investment and the premium for non-smooth consumption across time and states, in addition to the scarcity of young agents with high learning ability and endowment. Even if the share of return on intangible capital in gross domestic income ($r_t K_t / GDI_t$) is relatively stable in theory, the share of measured "profit" in national income is an increasing function of the degree of limited commitment and the volatility of idiosyncratic productivity shock in intangible capital investment, as is shown in Figures 5 and 6.

3 Full Model

In the Basic model, managers never move to different firms from those at which they received training. In data, some managers move to different firms or start new firms. In Japan, Kawaguchi and Ueno (2013) documented that the mean tenure at age 40 declined from 15 years for the male of birth cohort 1944-49 to 12 years for the male of birth cohort 1970-81 (who typically entered into the job market after the stagnation started in 1992), according to the Employment Status Survey. The fall in the mean tenure is significant even after controlling the effect of longer education for more recent cohorts. To explain why workers move between firms and why the job tenure declined recently in Japan, we extend the model to allow shocks to quality of match between trainees and firms. We also take into account the effect of aggregate shock more systematically.

3.1 Framework

We extend the framework of the basic model to an economy with aggregate uncertainty and match quality shock between trainees and firms. In this economy, the exogenous aggregate state s could affect the common aggregate productivity of goods production, A_s, the distribution parameter of endowment, ω_s, and the specificity of intangible capital, θ_s. The state follows a Markov process. From the state s of this period, the next-period state, s', follows distribution $\Pi(s'|s)$. As in the basic model, the endogenous state variables include aggregate intangible capital stock, K, and routine labor supply from the old generation, L^o. $S \equiv (K, L^o, s)$, summarizes the aggregate state variables. For convenience, we denote the exogenous state at aggregate state S to be s and denote the exogenous state at aggregate state S' to be s'.

Because we focus on recursive equilibrium, prices depend on the current state variables. At state S, the wage rate is denoted w_S, the rate of return on intangible capital is denoted r_S. We allow Arrow securities to be traded in the competitive financial market, each promising one final good contingent on the aggregate state next period, s'. The price of an Arrow security for state s' in the next period given today's state S is denoted by $q_{s'|S}$.

Match quality shock is an idiosyncratic productivity shock to a manager-firm pair. Given the expected intangible capital output, k^+, the realized intangibles of a trainee is

$$k^+_{z,\zeta} = \zeta z k^+. \tag{30}$$

ζ is a match quality shock, which is idiosyncratic to the trainee-firm pair. z is a productivity shock idiosyncratic to the trainee but common across all firms, as in the basic model. ζ and z are independent from each other, with ζ following marginal distribution function $\Phi_\zeta(\cdot)$ and

z following marginal distribution $\Phi_z(\cdot)$, and the joint distribution of ζ and z, $\Phi(\cdot,\cdot)$ equals $\Phi_\zeta(\cdot)\Phi_z(\cdot)$. Both marginal distributions have full support on $(0,\infty)$ and expected value 1.

When a manager's match quality with the current firm, ζ, is very low, a manager may be better off by moving from the current firm to a new firm. When the manager moves to a new firm when old, she loses θ fraction of general skills but get a new draw $\widetilde{\zeta}$ of match quality with new firm, which we assume is independent from her match quality with the current firm. The manager cannot move more than once in our model with two-period lifetime. We allow recruiting firms to provide insurance against match quality shocks to recruit managers. Through perfect competition between recruiting firms, the compensation for the recruited manager with $zk_S^+(\kappa,e)$ intangible before moving would equal the expected return of her intangible capital as:

$$r_{S'}E\left(\widetilde{\zeta}\right)(1-\theta_s)zk_S^+(\kappa,e) = r_{S'}(1-\theta_s)zk_S^+(\kappa,e).$$

Because the match quality is publicly observable and insurable, the manager's compensation does not depend upon the match quality while it depends upon idiosyncratic productivity and aggregate state as

$$c^o = c^o_{z,S'|S}(\kappa,e).$$

Since the manager is free to move to a new firm, the compensation must satisfy

$$c^o_{z,S'|S}(\kappa,e) \geq r_{S'}(1-\theta_s)zk_S^+(\kappa,e). \tag{31}$$

Because the firm can precommit to the contract even for the agent who leaves, we allow the firm to provide her the compensation which exceeds the outside option $r_{S'}(1-\theta_s)zk_S^+(\kappa,e)$ by paying the gap $c^o_{z,S'|S}(\kappa,e) - r_{S'}(1-\theta_s)zk_S^+(\kappa,e)$ as the severance payment.[17]

[17] Alternatively, if the manager has the right to stay in the same firm and has all the bargaining power against the firm, the firm ends up paying the severance payment $c^o_{z,S'|S}(\kappa,e) - r_{S'}(1-\theta_s)zk_S^+(\kappa,e)$ to persuade the manager with low quality match $\zeta < 1-\theta_s$ to leave. If, instead, the firm has all the bargaining power and cannot precommit to provide the severance payment for managers who leave the firm, then the compensation for the leaving manager must equal to the outside option of the competitive rate as:

$$c^o_{z,S'|S}(\kappa,e) = r_{S'}(1-\theta_s)zk_S^+(\kappa,e).$$

Then the competitive equilibrium lacks the coordination and is no longer constrained efficient (which we will show below).

To recruit a trainee of type (κ, e) $(n_S(\kappa, e) > 0)$, the compensation package must satisfy

$$\ln c^y_S(\kappa, e) + \beta \int \int \ln c^o_{z,S'|S}(\kappa, e) d\Phi d\Pi(s'|s) \geq V_S(\kappa, e). \tag{32}$$

The present value of profit for the firm equals the sum of present gross profit plus the net receipts from recruiting young agents for future managers as:

$$\begin{aligned} &A_s (K^w_S)^\alpha L_S{}^{1-\alpha} - w_S L^w_S + \int \{-[c^y_S(\kappa, e) - e] \\ &+ \int q_{s'|S} \int \left[r_{S'} \max(\zeta, 1-\theta_s) \cdot z k^+_S(\kappa, e) - c^o_{z,S'|S}(\kappa, e)\right] d\Phi d\Pi(s'|s) \Big\} n_S(\kappa, e) dF_s. \end{aligned} \tag{33}$$

The net receipts from the next period (in the second line) is now contingent also on the aggregate state, S', and the match quality ζ. When $\zeta \geq 1 - \theta_s$, the firm retains the manager and receives $r_{S'} \zeta z k^+_S(\kappa, e) - c^o_{z,S'|S}(\kappa, e)$ in net. When $\zeta < 1 - \theta_s$, the firm dissolves the employment of the manger and may pay the severance payment $c^o_{z,S'|S}(\kappa, e) - r_{S'}(1-\theta_s) z k^+_S(\kappa, e)$.

The representative firm with K_t total intangible capital of current managers chooses final goods production K^w_S, L_S, L^w_S and the recruitment package of future managers $\{n_S, \tilde{k}_S, h_S, k^+_S, c^y_S, c^o_{z,S'|S}(\kappa, e)\}$ to maximize the present value of profit (33), subject to the constraints of intangible capital,

$$K_S = K^w_S + \int \widetilde{k}_S(\kappa, e) n_S(\kappa, e) dF_S, \tag{34}$$

labor,

$$L_S = L^w_S + \int [1 - h_S(\kappa, e)] n_S(\kappa, e) dF_S, \tag{35}$$

participation (32) and incentive (31).

For young workers of type (κ, e) not trained by current managers, their lifetime value, $V^w_S(\kappa, e)$, is the solution of the following consumption-saving problem:

$$\begin{gathered} V^w_S(\kappa, e) = \max_{c^y_S, c^o_{S'} \geq 0} \left[\ln c^y_S + \beta \int \ln c^o_{S'|S} d\Pi(s'|s)\right] \\ \text{subject to } c^y_S + \sum_{s'} q_{s'|S} c^o_{S'|S} = e + w_S + \sum_{s'} q_{s'|S} w_{S'}. \end{gathered}$$

The outside option of the trainee of type (κ, e) is given by the maximum between $V^w_S(\kappa, e)$ and the highest expected utility offered by various firms as

$$V_S(\kappa, e) = \max \left\{ V^w_S(\kappa, e),\ \max_{\text{all firms}} \left[\ln c^y_S(\kappa, e) + \beta \int \int \ln c^o_{z,S'|S}(\kappa, e) d\Phi d\Pi(s'|s)\right] \right\}.$$

Definition 2. A stochastic recursive equilibrium is firms' policies K_S^w, L_S^w, $\{n_S, c_S^y, h_S, \widetilde{k}_S, k_S^+, c_{z,S'|S}^o, V_S\}(\kappa, e)$, routine worker's consumption plan $\left\{c_S^y, c_{S'|S}^o\right\}(\kappa, e)$, return of intangibles r_S, bond price $q_{s'|S}$, wage rate w_S, as functions of aggregate state $S \equiv (K, L^o, s)$ such that

a) Given prices $\left(r_S, q_{s'|S}, w_S\right)$ and the outside option $V_S(\kappa, e)$, firms' policy functions solve their problem;

b) Given firms' policies, outside option of the trainee is consistent with equilibrium (10), and labor and financial markets clear,

$$L_S^w = L_S^o + L_{S'}^o,$$

$$\int_{\Theta_S}\int c_{z,S'|S}^o(\kappa, e)d\Phi dF_S + \int_{\Theta_S^c} c_{S'|S}^o(\kappa, e)dF_S \tag{36}$$

$$= \int_{\Theta_S}\int r_{S'}z\max\left(\zeta, 1-\theta_s\right)k_S^+(\kappa, e)d\Phi dF_S + \int_{\Theta_S^c} w_{S'}(\kappa, e)dF_S, \forall S'$$

$$\text{where } \Theta_S = \{(\kappa, e) : n_S(\kappa, e) > 0\}; \tag{37}$$

c) $K_{S'}$ and $L_{S'}^o$ follow the laws of motion

$$L_{S'}^o = \int[1 - n_S(\kappa, e)]dF_S, \tag{38}$$

$$K_{S'} = \int_{\Theta_S}\int \max\left(\zeta, 1-\theta_s\right)zk_S^+(\kappa, e)n_S(\kappa, e)d\Phi dF_s. \tag{39}$$

3.2 Competitive Equilibrium with Match Quality Shocks

As before, we can use the investment cost function $\varphi_S(k^+;\kappa)$ to analyze the equivalent dual problem in which a type-(κ, e) trainee chooses an optimal consumption and investment profile subject to the budget constraint and the incentive constraints as:

$$V_S^m(\kappa, e) = \max_{c_S^y(\kappa,e), k_S^+(\kappa,e), c_{z,S'|S}^o(\kappa,e)}\left[\ln c_S^y(\kappa, e) + \beta\int \ln c_{z,S'|S}^o(\kappa, e)d\Phi d\Pi\right]$$

$$\text{s.t. } c_S^y(\kappa, e) + \varphi_S(k_S^+(\kappa, e);\kappa) + \int q_{s'|S}c_{z,S'|S}^o\left(\kappa, e\right)d\Phi d\Pi$$

$$= e + w_S + \int\int q_{s'|S}r_{S'}z\max\left(\zeta, 1-\theta_s\right)k_S^+(\kappa, e)d\Phi d\Pi,$$

$$c^o_{z,S'|S}(\kappa, e) \geq r_{S'}(1-\theta_s) z k^+_S(\kappa, e).$$

Similar to the Basic Model, we can show that

$$c^o_{z,S'|S}(\kappa, e) = (1-\theta_s) r_{S'} \max\left[z,\ z^*_{S'|S}(\kappa, e)\right] k^+_S(\kappa, e), \tag{40}$$

$$\frac{\beta}{q_{s'|S}} c^y_S(\kappa, e) = (1-\theta_s) r_{S'} z^*_{S'|S}(\kappa, e)\, k^+_S(\kappa, e), \tag{41}$$

$$\varphi'_S(k^+_S(\kappa, e); \kappa) = \int r_{S'} Q_{s'|S}\left(z^*_{S'|S}(\kappa, e), \theta_s\right) d\Pi, \text{ where} \tag{42}$$

$$Q_{S'|S}\left(z^*_{S'|S}(\kappa, e), \theta_s\right) \equiv q_{s'|S}\left\{\int \max(\zeta, 1-\theta_s)\, d\Phi_\zeta - (1-\theta_s)\int_{z^*_{S'|S}(\kappa,e)}^{\infty}\left[z - z^*_{S'|S}(\kappa, e)\right] d\Phi_z\right\},$$

$$(1+\beta) c^y_S(\kappa, e) + \varphi_S(k^+_S(\kappa, e); \kappa) = e + w_S + \left[\int r_{S'} Q_{S'|S}\left(z^*_{S'|S}(\kappa, e), \theta_s\right) d\Pi\right] k^+_S(\kappa, e). \tag{43}$$

Equation (40) implies that the consumption when old equals a larger value of the minimum consumption or the consumption with which the incentive constraint binds, and (41) implies the minimum consumption satisfies the Euler equation under complete market for the aggregate state. Equation (42) is the marginal condition for intangible investment, which takes into account the match quality shock and the partially insurable idiosyncratic productivity shock, and (43) is the lifetime budget constraint for type (κ, e) who becomes a trainee. From the above four equations, we can solve for four functions $c^o_{z,S'|S}(\kappa, e)$, $z^*_{S'|S}(\kappa, e)$, $k^+_S(\kappa, e)$ and $c^y_S(\kappa, e)$.

The value of type-(κ, e) trainee is

$$V^m_S(\kappa, e) = (1+\beta)\ln\left\{e + w_S + k^+_S(\kappa, e)\int r_{S'} Q_{S'|S}\left(z^*_{S'|S}(\kappa, e), \theta_s\right) d\Pi - \varphi_S(k^+_S(\kappa, e); \kappa)\right\}$$
$$+ \beta\int\left\{\int_{z^*_{S'|S}(\kappa,e)}^{\infty}\left[\ln z - \ln z^*_{S'|S}(\kappa, e)\right] d\Phi_z + \ln\left(\frac{\beta}{q_{s'|S}}\right)\right\} d\Pi - (1+\beta)\ln(1+\beta).$$

The value of type-(κ, e) routine worker is

$$V^w_S(\kappa, e) = (1+\beta)\ln\left(\frac{e + w_S + \int q_{s'|S} w_{S'} d\Pi}{1+\beta}\right) + \beta\int \ln\left(\frac{\beta}{q_{s'|S}}\right) d\Pi.$$

Thus type-(κ, e) young agent chooses to become trainee, $(\kappa, e) \in \Theta_S$, if and only if

$$(1+\beta)\ln\left\{\frac{e + w_S + k^+_S(\kappa, e)\int r_{S'} Q_{S'|S}\left(z^*_{S'|S}(\kappa, e), \theta_s\right) d\Pi - \varphi_S(k^+_S(\kappa, e); \kappa)}{e + w_S + \int q_{s'|S} w_{S'} d\Pi}\right\}$$
$$+ \beta\int\int_{z^*_{S'|S}(\kappa,e)}^{\infty}\left[\ln z - \ln z^*_{S'|S}(\kappa, e)\right] d\Phi_z d\Pi > 0. \tag{44}$$

The law of motion for old routine workers in the next period is

$$L^o_{S'} = \int_{\Theta^c_S} dF_S, \tag{45}$$

and the law of motion of the intangible capital of the next period is given by (39). Present labor market and intangible capital market are given by

$$L_S = L^o_S + L^o_{S'} + \int_{\Theta_S} [1 - h_S(\kappa, e)] dF_S, \tag{46}$$

$$K_S = K^w_S + \int_{\Theta_S} \widetilde{k}_S(\kappa, e) dF_S, \tag{47}$$

where

$$h_S(\kappa, e) = \frac{\partial}{\partial w_S} \varphi_S(k^+_S(\kappa, e); \kappa), \text{ and } \widetilde{k}_S(\kappa, e) = \frac{\partial}{\partial r_S} \varphi_S(k^+_S(\kappa, e); \kappa),$$

by the property of the cost function.

The contingent bond market equilibrium is given by (36), and the rate of return on intangible and wage are

$$r_S = \alpha A_s \left(\frac{L_S}{K^w_S} \right)^{1-\alpha}, \tag{48}$$

$$w_S = (1 - \alpha) A_s \left(\frac{K^w_S}{L_S} \right)^{\alpha}. \tag{49}$$

The recursive equilibrium is defined as $\left(c^y_S, c^o_{z,S'|S}, z^*_{S'|S}, k^+_{S'|S} \right) (\kappa, e), \Theta_S, L_S, K^w_S, r_S, w_S, q_{s'|S}, L^o_{S'}, K_{S'}$ as functions of the state variable $S = (s, K, L^o)$ which satisfy twelve equations $(40) - (49)$.

As in Basic Model, we show in Section B.1 of the Appendix that we can find Pareto weights in which the solution of the social planner's problem corresponds to the competitive equilibrium.

3.3 Calibration

4 Conclusion

Our paper offers a tractable framework to study how intangible capital accumulation within firms interacts with income and consumption of managers at the micro level and aggregate productivity at the macro level. We show that when there is a negative shock to endowment or degree of firm specificity of intangible capital, labor productivity falls and income becomes more unequal persistently as we observe in developed countries in recent decades.

Two particular features of intangible capital (managerial skill) contribute to the interaction. First, intangible capital is not directly transferrable and needs to be accumulated through costly training on the job. Second, intangible capital is hard to pledge as collateral because future managers cannot be forced to stay and work in the same firm. This makes it harder for future managers to smooth consumption over lifetime and across states, and in turn reduces intangible capital accumulation and increases income inequality because intangible capital accumulation must be compensated for the induced non-smooth consumption profile.

The limited commitment becomes severer when intangible capital is less firm-specific and managers are consequently more mobile. Exploring the policy implications of the lower firm-specificity of human capital and the higher mobility of skilled workers is a topic for the future research.

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A Basic Model

A.1 Training Cost function

Let us define the minimum cost of training young agent of type (κ, e) to acquire the expected level intangible capital k^+ as

$$\varphi_t(k^+;\kappa) = \min_{h,\tilde{k}} \left[w_t h + r_t \tilde{k},\right]$$
$$\text{subject to } k^+ = (1/b)\tilde{k}^\eta (h\kappa)^{1-\eta} \text{ and } 0 \leq h \leq 1.$$

The cost of acquiring intangibles is the opportunity cost for trainees to allocate h units of time and for current managers to allocate $\tilde{k}$ units of intangibles. When $0 < h < 1$, the factor price ratio equals the ratio of marginal product as

$$\frac{w_t}{r_t} = \frac{1-\eta}{\eta}\frac{\tilde{k}}{h}.$$

Thus

$$\varphi_t(k^+;\kappa) = \left(\frac{r}{\eta}\right)^\eta \left(\frac{w}{(1-\eta)\kappa}\right)^{1-\eta} bk^+.$$

From these two equation, we can verify that $h < 1$ if and only if

$$k^+ < \left(\frac{\eta}{1-\eta}\frac{w}{r}\right)^\eta \frac{\kappa^{1-\eta}}{b} \equiv \overline{k}(\kappa).$$

If $k^+ > \overline{k}(\kappa)$, then we learn $h = 1$ so that

$$\tilde{k} = \left[bk^+ / \left(\kappa^{1-\eta}\right)\right]^{\frac{1}{\eta}}$$

and

$$\varphi_t(k^+;\kappa) = w + r\left[bk^+ / \left(\kappa^{1-\eta}\right)\right]^{\frac{1}{\eta}}$$

Therefore we learn

$$\varphi_t(k^+;\kappa) = \begin{cases} \left(\frac{r}{\eta}\right)^\eta \left(\frac{w}{(1-\eta)\kappa}\right)^{1-\eta} bk^+, \text{ for } k^+ < \left(\frac{\eta}{1-\eta}\frac{w}{r}\right)^\eta \frac{\kappa^{1-\eta}}{b} \equiv \overline{k}(\kappa), \\ w + r\left[bk^+ / \left(\kappa^{1-\eta}\right)\right]^{\frac{1}{\eta}}, \text{ for } k^+ \geq \overline{k}(\kappa). \end{cases} \tag{A.1}$$

A.2 Equilibrium Analysis with full commitment.

Under full commitment, the trainee solves

$$V_t^m(\kappa,e) = \max_{c_t^y(\kappa,e),k^+(\kappa,e),c_z^o(\kappa,e)} \left[\ln c_t^y(\kappa,e) + \beta \int \ln c_{z,t+1}^o(\kappa,e) d\Phi(z)\right]$$
$$\text{s.t. } c_t^y(\kappa,e) + \varphi_t(k^+(\kappa,e);\kappa) + q_t \int c_{z,t+1}^o(\kappa,e)\, d\Phi = e + w_t + q_t r_{t+1} k_t^+(\kappa,e).$$

Using the Lagrangian,

$$\mathcal{L} = \ln c_t^y(\kappa,e) + \beta \int \ln c_{z,t+1}^o(\kappa,e) d\Phi(z)$$
$$+ \lambda_t \left[e + w_t + q_t r_{t+1} k_t^+(\kappa,e) - c_t^y(\kappa,e) - \varphi_t(k^{+(\kappa,e)};\kappa) - q_t \int c_{z,t+1}^o(\kappa,e)\, d\Phi\right],$$

we get the first order conditions as

$$\frac{1}{c_t^y(\kappa,e)} = \lambda_t,$$
$$\frac{\beta}{c_{z,t+1}^o(\kappa,e)} = \lambda_t q_t,$$
$$\varphi_t'(k^{+(\kappa,e)};\kappa) = q_t r_{t+1}.$$

Because $\varphi_t'(k^+(\kappa,e);\kappa)$ is strictly decreasing function of κ, we learn $h_t(\kappa,e) = 1$ and $q_t r_{t+1} = \varphi_t'(k_t^+(\kappa,e);\kappa) = \frac{1}{\eta} r_t b^{\frac{1}{\eta}} \left(\frac{k_t^+(\kappa,e)}{\kappa}\right)^{\frac{1-\eta}{\eta}}$, or

$$k_t^+(\kappa,e) = a_t^* \kappa, \text{ where} \tag{A.2}$$
$$a_t^* = \left(\frac{\eta q_t r_{t+1}}{b^{1/\eta} r_t}\right)^{\frac{\eta}{1-\eta}}. \tag{A.3}$$

and

$$\tilde{k}_t(\kappa,e) = \left[b k_t^+(\kappa,e) / \left(\kappa^{1-\eta}\right)\right]^{\frac{1}{\eta}} = (b a_t^*)^{\frac{1}{\eta}} \kappa.$$

Then we learn

$$c_t^y(\kappa,e) = \frac{q_t}{\beta} c_{z,t+1}^o(\kappa,e) = \frac{1}{1+\beta}\left[e + (1-\eta) q_t r_{t+1} k_t^+(\kappa,e)\right], \tag{A.4}$$

and the expected utility is given by

$$V_t^m(\kappa,e) = (1+\beta)\ln\left(\frac{1}{1+\beta}\left[e + (1-\eta) q_t r_{t+1} k_t^+(\kappa,e)\right]\right) + \beta \ln\left(\frac{\beta}{q_t}\right).$$

From the routine worker's maximization, we get

$$c_t^y(\kappa,e) = \frac{q_t}{\beta} c_{t+1}^o(\kappa,e) = \frac{1}{1+\beta}\left[e + w_t + q_t w_{t+1}\right], \tag{A.5}$$

and the expected utility is given by

$$V_t^w(\kappa, e) = (1+\beta)\ln\left(\frac{1}{1+\beta}\left[e + w_t + q_t w_{t+1}\right]\right) + \beta \ln\left(\frac{\beta}{q_t}\right). \tag{A.6}$$

Therefore young agents become trainees if and only if

$$e + (1-\eta) q_t r_{t+1} k_t^+(\kappa, e) > e + w_t + q_t w_{t+1},$$

or

$$\kappa > \kappa_t^* \equiv \frac{w_t + q_t w_{t+1}}{(1-\eta) q_t r_{t+1} a_t^*}. \tag{A.7}$$

Let $F_t(\kappa, e) \equiv G_t(e) H_t(\kappa)$. Then we get

$$L_{t+1}^o = H_t(\kappa_t^*), \text{ and} \tag{A.8}$$

$$K_{t+1} = a_t^* \int_{\kappa_t^*}^{\kappa_H} \kappa dH_t(\kappa). \tag{A.9}$$

The market equilibrium for labor and intangible capital become

$$L_t = L_{t+1}^o + L_t^o, \tag{A.10}$$

$$K_t = K_t^w + \int_{\kappa_t^*}^{\kappa_H} \tilde{k}(\kappa, e)\, dH_t(\kappa) = K_t^w + (b a_t^*)^{\frac{1}{\eta}} \int_{\kappa_t^*}^{\kappa_H} \kappa dH_t(\kappa). \tag{A.11}$$

The equilibrium wage rate is,

$$w_t = A(1-\alpha)\left(\frac{K_t^w}{L_t}\right)^{\alpha}. \tag{A.12}$$

The rate of return on intangible capital r_t is

$$r_t = \alpha A_t \left(\frac{L_t}{K_t^w}\right)^{1-\alpha}, \tag{A.13}$$

as in the text.

Let S_t^y be the aggregate net worth of young generation at the end of period t. Because the net worth of the old generation equals zero at the end of period t, the market clearing implies

$$S_t^y = 0.$$

Let $\overline{e}_t$ be the aggregate (or average) endowment of young agents.

$$\overline{e}_t \equiv \int e dG_t(e).$$

Then the market clearing condition for aggregate net worth of young generation is

$$
\begin{aligned}
0 = S_t^y &= \bar{e}_t + w_t - \int_{\kappa_t^*}^{\kappa_H} \varphi_t(a_t^* \kappa; \kappa) dH_t(\kappa) - \int c_t^y(\kappa, e) dF_t(\kappa, e) \\
&= \bar{e}_t + w_t L_{t+1}^o - \eta q_t r_{t+1} K_{t+1} - \frac{1}{1+\beta} \left[\bar{e}_t + q_t r_{t+1}(1-\eta) K_{t+1} + (w_t + q_t w_{t+1}) L_{t+1}^o \right] \\
&= \frac{\beta}{1+\beta} \left(\bar{e}_t + w_t L_{t+1}^o \right) - \frac{q_t w_{t+1}}{1+\beta} L_{t+1}^o - \left(\eta + \frac{1-\eta}{1+\beta} \right) q_t r_{t+1} K_{t+1}. \qquad \text{(A.14)}
\end{aligned}
$$

The dynamic equilibrium of the aggregate economy under full commitment is given by nine endogenous variables $(w_t, r_t, q_t, a_t^*, \kappa_t^*, L_t, K_t^w, K_{t+1}, L_{t+1})$ as a function of the state variable $S_t = (K_t, L_t^o, s_t)$ which satisfies ten equations $(A.3), (A.7) - (A.14)$. Then all the individual choice $\{c_t^y(\kappa, e), c_{t+1}^o(\kappa, e), k_t^+(\kappa, e)\}$ are determined from $(A.2, A.4, A.5)$ as a function of aggregate state S_t and the individual characteristics (κ, e).

A.3 Equilibrium analysis with Limited Commitment

Now we complement the description of equilibrium analysis under binding limited commitment in Section **??**.

Choice of Firm and Manager

We solve for the dual problem of manager described by (20), (21) and (22). Using the Lagrangian

$$
\begin{aligned}
\mathcal{L} &= \ln c_t^y(\kappa, e) + \beta \int c_{z,t+1}^o(\kappa, e) d\Phi \\
&+ \lambda_t \left[e + w_t + q_t r_{t+1} k_t^+(\kappa, e) - \varphi_t \left(k_t^+(\kappa, e) \right) - c_t^y(\kappa, e) - q_t \int c_{z,t+1}^o(\kappa, e) d\Phi \right] \\
&+ \int \mu_{z,t+1} \left[c_{z,t+1}^o(\kappa, e) - (1-\theta) r_{t+1} z k_t^+(\kappa, e) \right] d\Phi,
\end{aligned}
$$

we get the first order conditions as

$$
\frac{1}{c_t^y(\kappa, e)} = \lambda_t,
$$

$$
\frac{\beta}{c_{z,t+1}^o(\kappa, e)} = \lambda_t q_t - \mu_{z,t+1},
$$

$$
\varphi_t' \left(k_t^+(\kappa, e) \right) = q_t r_{t+1} - \int \frac{\mu_{z,t+1}}{\lambda_t} (1-\theta) r_{t+1} z d\Phi.
$$

From the first two, we learn

$$
c_{z,t+1}^o = \begin{cases} (1-\theta) r_{t+1} z k_t^+(\kappa, e), \text{ for } z > z_t^*(\kappa, e) \\ (1-\theta) r_{t+1} z^*(\kappa, e) k_t^+(\kappa, e), \text{ for } z \leq z_t^*(\kappa, e) \end{cases}, \qquad \text{(A.15)}
$$

where

$$(1-\theta)r_{t+1}z_t^*(\kappa,e)k_t^+(\kappa,e)=\frac{\beta}{q_t}c_t^y(\kappa,e), \tag{A.16}$$

and

$$\frac{\mu_{z,t+1}}{\lambda_t}=q_t-\frac{\beta c_t^y(\kappa,e)}{c_{z,t+1}^o(\kappa,e)}=q_t\left[1-\frac{z_t^*(\kappa,e)}{z}\right].$$

Then from the third equation, we learn

$$\begin{aligned}\varphi_t'(k_t^+(\kappa,e),\kappa)&=r_{t+1}Q_t\left(z_t^*(\kappa,e)\right),\ \text{where} \\ Q_t\left(z_t^*(\kappa,e)\right)&\equiv q_t\left[1-(1-\theta)\int_{z^*(\kappa,e)}^{\infty}[z-z_t^*(\kappa,e)]d\Phi(z)\right],\end{aligned} \tag{A.17}$$

as in text.

Then from (21), we get

$$\begin{aligned}&e+w_t+q_tr_{t+1}k_t^+(\kappa,e)-\varphi_t\left(k_t^+(\kappa,e);\kappa\right) \\ &=(1+\beta)c_t^y(\kappa,e)+q_t\int_{z_t^*(\kappa,e)}^{\infty}(1-\theta)r_{t+1}[z-z_t^*(\kappa,e)]d\Phi k_t^+(\kappa,e),\end{aligned}$$

or

$$(1+\beta)\,c^y(\kappa,e)+\varphi_t(k_t^+(\kappa,e),\kappa)=e+w_t+r_{t+1}Q_t\left(z_t^*(\kappa,e)\right)k_t^+(\kappa,e). \tag{A.18}$$

as in the text. $c_t^y(\kappa,e)$, $z_t^*(\kappa,e)$ and $k_t^+(\kappa,e)$ solve $(A.16)$, $(A.17)$ and $(A.18)$ for given prices. Then $c_{z,t+1}^o(\kappa,e)$ is given by $(A.15)$.

The discounted expected utility of a type-(κ,e) trainee is

$$\begin{aligned}V_t^m(\kappa,e)&=\ln c_t^y(\kappa,e)+\beta\int\ln c_{z,t+1}^o(\kappa,e)d\Phi \\ &=(1+\beta)\ln c_t^y(\kappa,e)+\beta\ln\left(\frac{\beta}{q_t}\right)+\beta\int_{z_t^*(\kappa,e)}^{\infty}[\ln z-\ln z_t^*(\kappa,e)]d\Phi.\end{aligned}$$

The routine worker's maximization is the same as with full commitment as in $(A.5, A.6)$. Thus type-(κ,e) agent chooses to become a trainee, or $(\kappa,e)\in\Theta_t$, if and only if $V_t^m>V_t^w$, or

$$\begin{aligned}&(1+\beta)\ln\left[e+w_t+r_{t+1}Q_t\left(z_t^*(\kappa,e)\right)k_t^+(\kappa,e)-\varphi_t(k_t^+(\kappa,e);\kappa)\right]+\beta\int_{z_t^*(\kappa,e)}^{\infty}[\ln z-\ln z_t^*(\kappa,e)]d\Phi \\ &>(1+\beta)\ln\left(e+w_t+q_tw_{t+1}\right).\end{aligned} \tag{A.19}$$

For the case of $k_t^+(\kappa,e)<\overline{k}(\kappa)$ in $(A.1)$, $(A.17)$ becomes

$$\left(\frac{r_t}{\eta}\right)^{\eta}\left(\frac{w_t}{(1-\eta)\kappa}\right)^{1-\eta}b=r_{t+1}Q_t\left(z_t^*(\kappa,e)\right),$$

which uniquely determines $z_t^*(\kappa, e)$. Because the LHS is a decreasing function of κ and the RHS is an increasing function of $z_t^*(\kappa, e)$, we learn

$$\frac{\partial}{\partial \kappa} z_t^*(\kappa, e) < 0 \text{ and } \frac{\partial}{\partial e} z_t^*(\kappa, e) = 0$$

Also from $(A.18)$ and $(A.16)$we learn

$$c_t^y(\kappa, e) = \frac{e + w_t}{1 + \beta} = \frac{q_t}{\beta}(1 - \theta) r_{t+1} z_t^*(\kappa, e) k_t^+(\kappa, e).$$

Thus

$$k_t^+(\kappa, e) = \frac{\beta}{(1 + \beta)(1 - \theta)} \frac{e + w_t}{q_t r_{t+1} z_t^*(\kappa, e)},$$

which implies

$$\frac{\partial}{\partial \kappa} k_t^+(\kappa, e) > 0, \; \frac{\partial}{\partial e} k_t^+(\kappa, e) > 0.$$

Also we learn $(\kappa, e) \in \Theta_t$, if and only if

$$(1 + \beta) \ln(e + w_t) + \beta \int_{z_t^*(\kappa,e)}^{\infty} [\ln z - \ln z_t^*(\kappa, e)] d\Phi > (1 + \beta) \ln (e + w_t + q_t w_{t+1}).$$

Thus young agent chooses to become a manager if and only if

$$(\kappa, e) \in \Theta_t \equiv \{(\kappa, e) : \kappa > \kappa_t^*(e)\},$$

where $\kappa_t^*(e)$ solves

$$(1 + \beta) \ln \left(\frac{e + w_t}{e + w_t + q_t w_{t+1}} \right) + \beta \int_{z_t^*(\kappa^*,e)}^{\infty} [\ln z - \ln z_t^*(\kappa^*, e)] d\Phi = 0.$$

Because $\frac{e+w_t}{e+w_t+q_t w_{t+1}}$ is an increasing function of e and $z_t^*(\kappa, e)$ is a decreasing function of κ, we learn

$$\kappa_t^{*\prime}(e) \leq 0.$$

For the case of $k_t^+(\kappa, e) > \overline{k}(\kappa)$ in $(A.1)$, $(A.17)$ becomes

$$\frac{1}{\eta} r_t b^{\frac{1}{\eta}} \left(\frac{k_t^+(\kappa, e)}{\kappa} \right)^{\frac{1-\eta}{\eta}} = r_{t+1} Q_t \left(z_t^*(\kappa, e) \right),$$

or

$$k_t^+(\kappa, e) = a_t^{**}(z_t^*(\kappa, e)) \cdot \kappa, \text{ where}$$

$$a_t^{**}(z_t^*(\kappa, e)) \equiv \left[\frac{\eta r_{t+1} Q_t \left(z_t^*(\kappa, e) \right)}{b^{\frac{1}{\eta}} r_t} \right]^{\frac{\eta}{1-\eta}}.$$

Also from $(A.18)$ and $(A.16)$, we learn

$$c_t^y(\kappa,e) = \frac{e + (1-\eta)r_{t+1}Q_t\left(z_t^*(\kappa,e)\right)a_t^{**}(z_t^*(\kappa,e))\kappa}{1+\beta}$$
$$= \frac{q_t}{\beta}(1-\theta)r_{t+1}z_t^*(\kappa,e)a_t^{**}(z_t^*(\kappa,e))\kappa.$$

We can solve this equation with respect to $z_t^*(\kappa,e)$. Then we learn

$$\frac{\partial}{\partial\kappa}z_t^*(\kappa,e) < 0,\ \frac{\partial}{\partial e}z_t^*(\kappa,e) > 0,$$
$$\frac{\partial}{\partial\kappa}k_t^+(\kappa,e) > 0,\ \frac{\partial}{\partial e}k_t^+(\kappa,e) > 0,$$
$$\frac{\partial}{\partial\kappa}\left(\frac{k_t^+(\kappa,e)}{\kappa}\right) < 0,\ \frac{\partial}{\partial e}\left(\frac{k_t^+(\kappa,e)}{\kappa}\right) > 0.$$

Also we learn $(\kappa,e) \in \Theta_t$, if and only if

$$(1+\beta)\ln\left[\frac{e + (1-\eta)r_{t+1}Q_t\left(z_t^*(\kappa,e)\right)a_t^{**}(z_t^*(\kappa,e))\kappa}{e + w_t + q_t w_{t+1}}\right] + \beta\int_{z_t^*(\kappa,e)}^{\infty}[\ln z - \ln z_t^*(\kappa,e)]d\Phi > 0.$$

Market clearing condition

As before, the endogenous state variables for the aggregate economy are old routine workers and intangible capital stock (L_t^o, K_t). Aggregate labor and intangible capital stock of the next period are:

$$L_{t+1}^o = \int_{(\kappa,e)\notin\Theta_t(\kappa,e)} dF_t(\kappa,e). \tag{A.20}$$

$$K_{t+1} = \int_{(\kappa,e)\in\Theta_t(\kappa,e)} k_t^+(\kappa,e)dF_t(\kappa,e). \tag{A.21}$$

The market clearing conditions for labor and intangible capital are

$$L_t = L_t^o + L_{t+1}^o + \int_{(\kappa,e)\in\Theta_t(\kappa,e)} h_t(\kappa,e)dF_t(\kappa,e), \tag{A.22}$$

$$K_t = K_t^w + \int_{(\kappa,e)\in\Theta_t(\kappa,e)} \widetilde{k}_t(\kappa,e)dF_t(\kappa,e), \tag{A.23}$$

where

$$h_t(\kappa,e) = \frac{\partial}{\partial w_t}\varphi_t\left(k_t^+(\kappa,e);\kappa\right),\ \text{and}\ \widetilde{k}_t(\kappa,e) = \frac{\partial}{\partial r_t}\varphi_t\left(k_t^+(\kappa,e);\kappa\right).$$

The wage rate and rate of return on intangible satisfy

$$w_t = (1-\alpha)A_t\left(\frac{K_t^w}{L_t}\right)^{\alpha}, \tag{A.24}$$

$$r_t = \alpha A_t\left(\frac{L_t}{K_t^w}\right)^{1-\alpha}. \tag{A.25}$$

The market clearing condition of funds is that the net worth of young agents at the end of date t equals zero, or

$$\begin{aligned} 0 &= S_t^y \qquad\qquad (\text{A.26}) \\ &= \int e dF_t + w_t - \int_{\Theta_t} \varphi_t(k_t^+(\kappa,e);\kappa) dF_t(\kappa,e) - \int c_t^y(\kappa,e) dF_t(\kappa,e). \end{aligned}$$

The dynamic equilibrium of the aggregate economy under limited commitment is given by four individual choice function $\{c_t^y, k_t^+, z_t^*, c_{z,t+1}^o\}(\kappa,e)$, one set Θ_t and seven endogenous aggregate variables $(w_t, r_t, q_t, L_t, K_t^w, K_{t+1}, L_{t+1}^o)$ as a function of the state variable $S_t = (K_t, L_t, s_t)$ which satisfies twelve equations $(A.15) - (A.26)$.

A.3.1 Constrained Efficiency of Competitive Equilibrium

The Lagrangian for the social planner's problem for the Basic Model is given by

$$\begin{aligned} \mathcal{L} = & \int \gamma_{t-1}(\kappa,e) \int \ln c_{z,t}^o(\kappa,e) d\Phi dF + \sum_{\tau=t}^{\infty} \int \gamma_\tau(\kappa,e) \left[\ln c_\tau^y(\kappa,e) + \beta \int \ln c_{z,\tau+1}^o(\kappa,e) d\Phi \right] dF \\ & + \sum_{\tau=t}^{\infty} \lambda_\tau \left[\bar{e}_t + A_\tau (K_\tau^w)^\alpha (L_\tau)^{1-\alpha} - \int c_\tau^y(\kappa,e) dF - \int\int c_{z,\tau}^o(\kappa,e) d\Phi dF \right] \\ & + \lambda_t w_t \left[L_t^o + \int (1 - h_t(\kappa,e)) dF - L_t \right] + \lambda_t r_t \left[K_t - K_t^w - \int \tilde{k}_t(\kappa,e) dF \right] \\ & + \sum_{\tau=t+1}^{\infty} \lambda_\tau w_\tau \left\{ \int (1 - h_\tau(\kappa,e)) dF - L_\tau \right\} \\ & + \sum_{\tau=t+1}^{\infty} \lambda_\tau r_\tau \left\{ \int \frac{1}{b} \left[\tilde{k}_{\tau-1}(\kappa,e) \right]^\eta (\kappa h_{\tau-1}(\kappa,e))^{1-\eta} dF - K_\tau^w - \int \tilde{k}_\tau(\kappa,e) dF \right\} \\ & + \sum_{\tau=t+1}^{\infty} \lambda_\tau \int \mu_{z,\tau}(\kappa,e) \left\{ c_{z,\tau}^o(\kappa,e) - r_\tau z(1-\theta) \frac{1}{b} \left[\tilde{k}_{\tau-1}(\kappa,e) \right]^\eta (\kappa h_{\tau-1}(\kappa,e))^{1-\eta} \right\} d\Phi \mathbb{I}_{h_t(\kappa,e)>0} dF. \end{aligned}$$

Using the training cost function (16), we can rewrite the above Lagrangian as

$$\begin{aligned} \mathcal{L} = & \int \gamma_{t-1}(\kappa,e) \int \ln c_{zt}^o(\kappa,e) d\Phi dF + \sum_{\tau=t}^{\infty} \lambda_\tau \left[A_\tau (K_\tau^w)^\alpha (L_\tau)^{1-\alpha} - r_\tau K_\tau^w - w_\tau L_\tau \right] \\ & + \sum_{\tau=t}^{\infty} \int \left\{ \begin{array}{c} \gamma_\tau(\kappa,e) \left[\ln c_\tau^y(\kappa,e) + \beta \int \ln c_{z,\tau+1}^o(\kappa,e) d\Phi \right] \\ +\lambda_\tau [e + w_\tau - c_\tau^y(\kappa,e) - \varphi_\tau(k_\tau^+(\kappa,e);\kappa)] \\ +\lambda_{\tau+1} r_{\tau+1} k_\tau^+(\kappa,e) \int [1 - \mu_{z,\tau}(\kappa,e)(1-\theta) z] d\Phi \\ -\lambda_{\tau+1} \int c_{z,\tau+1}^o(\kappa,e) [1 - \mu_{z,\tau}(\kappa,e)] d\Phi \end{array} \right\} \mathbb{I}_{h_\tau(\kappa,e)>0} dF \\ & + \sum_{\tau=t}^{\infty} \int \left\{ \begin{array}{c} \gamma_t(\kappa,e) \left[\ln c_t^y(\kappa,e) + \beta \ln c_{\tau+1}^o(\kappa,e) \right] \\ +\lambda_\tau [e + w_\tau - c_\tau^y(\kappa,e)] + \lambda_{\tau+1} \left[w_{\tau+1} - c_{\tau+1}^o(\kappa,e) \right] \end{array} \right\} \mathbb{I}_{h_\tau(\kappa,e)=0} dF, \end{aligned}$$

using $c^o_{z,\tau+1}(\kappa,e) = c^o_{\tau+1}(\kappa,e)$ for $(\kappa,e)-$type agents with $h_\tau(\kappa,e)=0$.

Then the first order conditions for $c^y_t(\kappa,e), c^o_{z,t+1}(\kappa,e)$, and $k^+_t(\kappa,e)$ for (κ,e) type agents with $h_\tau(\kappa,e)>0$ become

$$\frac{\gamma_t(\kappa,e)}{c^y_t(\kappa,e)} = \lambda_t,$$
$$\frac{\beta\gamma_t(\kappa,e)}{c^o_{z,t+1}(\kappa,e)} = \lambda_{t+1}[1-\mu_{z,t+1}(\kappa,e)],$$
$$\varphi'_t\left(k^+_t(\kappa,e);\kappa\right) = \frac{\lambda_{t+1}}{\lambda_t} r_{t+1}\int\left[1-(1-\theta)z\mu_{z,t+1}(\kappa,e)\right]d\Phi.$$

Defining $q_t = \frac{\lambda_{t+1}}{\lambda_t}$, we get

$$c^o_{z,t+1}(\kappa,e) = \max\left[z, z^*_t(\kappa,e)\right](1-\theta)r_{t+1}k^+_t(\kappa,e), \text{ where} \tag{A.27}$$
$$z^*_t(\kappa,e)(1-\theta)r_{t+1}k^+_t(\kappa,e) = \frac{\beta}{q_t}c^y_t(\kappa,e), \tag{A.28}$$
$$\mu_{z,t+1}(\kappa,e) = 1-\frac{\beta c^y_t(\kappa,e)}{q_t c^o_{z,t+1}(\kappa,e)} = 1-\frac{z^*_t(\kappa,e)}{z}$$

$$\varphi'_t(k^+_t(\kappa,e),\kappa) = r_{t+1}Q_t\left(z^*_t(\kappa,e)\right), \text{ where} \tag{A.29}$$
$$Q_t\left(z^*_t(\kappa,e)\right) = q_t\left[1-(1-\theta)\int_{z^*(\kappa,e)}^{\infty}[z-z^*(\kappa,e)]d\Phi(z)\right].$$

We also can choose the Pareto weight $\gamma_t(\kappa,e)$ to satisfy:

$$c^y_t(\kappa,e)+\varphi_t\left(k^+_t(\kappa,e);\kappa\right)+q_t\int c^o_{z,\tau+1}(\kappa,e)d\Phi = e+w_t+q_t r_{t+1}k^+_t(\kappa,e).$$

Using the first order conditions, the last equation can be rewritten as

$$(1+\beta)c^y_t(\kappa,e)+\varphi_t\left(k^+_t(\kappa,e);\kappa\right) = e+w_t+r_{t+1}Q_t\left(z^*_t(\kappa,e)\right)k^+_t(\kappa,e) \tag{A.30}$$

These four equations are the same for with the conditions for type (κ,e) trainee in the competitive equilibrium, concerning $c^y_t(\kappa,e), c^o_{z,t+1}(\kappa,e)$, $z^*_t(\kappa,e)$ and $k^+_t(\kappa,e)$ for the same q_t, w_t, r_t and r_{t+1}.

For the first order conditions for $c^y_t(\kappa,e)$ and $c^o_{t+1}(\kappa,e)$ for (κ,e) type agents with $h_t(\kappa,e)=0$, we get

$$c^y_t(\kappa,e) = \frac{q_t}{\beta}c^o_{t+1}(\kappa,e) = \frac{1}{1+\beta}(e+w_t+q_t w_{t+1}).$$

Thus the first order condition for the occupational choice, $I_{h_t(\kappa,e)>0}$, we get $h_t(\kappa,e)>0$ iff

$$\left\{\ln c^y_t(\kappa,e)+\beta\int\ln c^o_{z,\tau+1}(\kappa,e)d\Phi\right\}_{h_t(\kappa,e)>0} > \left\{\ln c^y_t(\kappa,e)+\beta\ln c^o_{t+1}(\kappa,e)\right\}_{h_t(\kappa,e)=0},$$

or

$$
\begin{aligned}
&(1+\beta)\ln\left\{e+w_t+r_{t+1}Q_t\left(z_t^*(\kappa,e)\right)k_t^+(\kappa,e)-\varphi_t\left(k_t^+(\kappa,e);\kappa\right)\right\}\\
&+\beta\int_{z^*(\kappa,e)}^{\infty}[\ln z-\ln z^*(\kappa,e)]d\Phi(z)\\
&>(1+\beta)\ln\left(e+w_t+q_t w_{t+1}\right).
\end{aligned}
$$

This is the same condition with the occupational choice in the competitive equilibrium.

Also from the first order condition for L_t and K_t^w, we have

$$
\begin{aligned}
w_t &= (1-\alpha)A_t\left(\frac{K_t^w}{L_t}\right)^{\alpha}\\
r_t &= \alpha A_t\left(\frac{L_t}{K_t^w}\right)^{1-\alpha}, \text{ and}\\
A_t\left(K_t^w\right)^{\alpha}\left(L_t\right)^{1-\alpha} &= r_t K_t^w + w_t L_t.
\end{aligned}
$$

From the above all, we find the Pareto weight $\gamma_t(\kappa,e)$ with which the solution of social planner's problem corresponds to the competitive equilibrium under the same constraint of the limited commitment.

B Full Model

We solve for the dual problem of manager using the Lagrangian

$$
\begin{aligned}
\mathcal{L} = &\ln c_S^y(\kappa,e)+\beta\int \ln c_{z,S'|S}^o(\kappa,e)d\Phi d\Pi\\
&+\lambda_S\left\{\begin{array}{c} e+w_S-\ c_S^y(\kappa,e)-\varphi_S(k_S^+(\kappa,e);\kappa)\\ +\int\int q_{s'|S}r_{S'}[z\max\left(\zeta,1-\theta_s\right)k_S^+(\kappa,e)-c_{z,S'|S}^o\left(\kappa,e\right)]d\Phi d\Pi\end{array}\right\}\\
&+\int\int \mu_{z,S'|S}\left[r_{S'}(1-\theta_s)zk_S^+(\kappa,e)-c_{z,S'|S}^o(\kappa,e)\right]d\Phi d\Pi,
\end{aligned}
$$

The first order conditions are

$$
\frac{1}{c_S^y(\kappa,e)}=\lambda_S,
$$

$$
\frac{\beta}{c_{z,S'|S}^o\left(\kappa,e\right)}=\lambda_S q_{s'|S}-\mu_{z,S'|S},
$$

$$
\varphi_S'(k_S^+(\kappa,e);\kappa)=\int\int q_{s'|S}r_{S'}\left[z\max\left(\zeta,1-\theta_s\right)-z(1-\theta_s)\frac{\mu_{z,S'|S}}{\lambda_S}\right]d\Phi d\Pi.
$$

Thus we get

$$c^{o}_{z,S'|S}(\kappa,e) = (1-\theta_s) r_{S'} \max\left[z,\ z^{*}_{S'|S}(\kappa,e)\right] k^{+}_{S}(\kappa,e), \tag{B.1}$$

$$\frac{\beta}{q_{s'|S}} c^{y}_{S}(\kappa,e) = (1-\theta_s) r_{S'} z^{*}_{S'|S}(\kappa,e)\, k^{+}_{S}(\kappa,e), \tag{B.2}$$

$$\varphi'_S(k^{+}_{S}(\kappa,e);\kappa) = \int r_{S'} Q_{s'|S}\left(z^{*}_{S'|S}(\kappa,e), \theta_s\right) d\Pi, \text{ where} \tag{B.3}$$

$$Q_{S'|S}\left(z^{*}_{S'|S}(\kappa,e), \theta_s\right) \equiv q_{s'|S}\left[\int \max(\zeta, 1-\theta_s)\, d\Phi_\zeta - (1-\theta_s)\int_{z^{*}_{S'|S}(\kappa,e)}^{\infty}\left[z - z^{*}_{S'|S}(\kappa,e)\right] d\Phi_z\right].$$

From the budget constraint, we get

$$\begin{aligned} & e + w_S + \int q_{s'|S} r_{S'} [z \max(\zeta, 1-\theta_s)\, k^{+}_{S}(\kappa,e)] d\Phi d\Pi - \varphi_S(k^{+}_{S}(\kappa,e);\kappa) \\ = \ & c^{y}_{S}(\kappa,e) + \beta c^{y}_{S}(\kappa,e) + (1-\theta_s)\int q_{s'|S} r_{S'} \int_{z^{*}_{S'|S}(\kappa,e)}^{\infty}\left[z - z^{*}_{S'|S}(\kappa,e)\right] d\Phi_z d\Pi, \end{aligned}$$

or

$$(1+\beta) c^{y}_{S}(\kappa,e) + \varphi_S(k^{+}_{S}(\kappa,e);\kappa) = e + w_S + \left[\int r_{S'} Q_{S'|S}\left(z^{*}_{S'|S}(\kappa,e), \theta_s\right) d\Pi\right] k^{+}_{S}(\kappa,e). \tag{B.4}$$

The discounted expected utility of a type-(κ, e) trainee is

$$\begin{aligned} V^{m}_{S}(\kappa,e) &= (1+\beta)\ln c^{y}_{S}(\kappa,e) + \beta\int\left\{\int_{z^{*}_{S'|S}(\kappa,e)}^{\infty}[\ln z - \ln z^{*}_{S'|S}(\kappa,e)] d\Phi_z + \ln\left(\frac{\beta}{q_{s'|S}}\right)\right\} d\Pi \\ &= (1+\beta)\ln\left\{e + w_S + k^{+}_{S}(\kappa,e)\int r_{S'} Q_{S'|S}\left(z^{*}_{S'|S}(\kappa,e), \theta_s\right) d\Pi - \varphi_S(k^{+}_{S}(\kappa,e);\kappa)\right\} \\ &\quad + \beta\int\left[\int_{z^{*}_{S'|S}(\kappa,e)}^{\infty}\left[\ln z - \ln z^{*}_{S'|S}(\kappa,e)\right] d\Phi_z + \ln\left(\frac{\beta}{q_{s'|S}}\right)\right] d\Pi - (1+\beta)\ln(1+\beta). \end{aligned}$$

B.1 Constrained Efficiency of the Competitive Equilibrium

Define the history as

$$S^t = (S_0, S_1, ..., S_t),$$

and the probability measure of S^t as $\Pi\left(S^t\right)$. We consider the Pareto weight of type-(κ, e) agent as

$$\gamma_t(\kappa,e) = \gamma_{S^t}(\kappa,e).$$

The Lagrangian of the planner's problem at date 0 is given by:

$$
\begin{aligned}
\mathcal{L} = & \int \gamma_{S_{-1}}(\kappa,e) \int \ln c^o_{z,S_0}(\kappa,e) d\Phi dF \\
& + \sum_{t=0}^{\infty} \int \gamma_{S^t}(\kappa,e) \left[\ln c^y_{S^t}(\kappa,e) d\Pi\left(S^t\right) + \beta \int \ln c^o_{z,S^{t+1}}(\kappa,e) d\Phi d\Pi\left(S^{t+1}\right) \right] dF \\
& + \sum_{t=0}^{\infty} \lambda_{S^t} \left[\bar{e}_{S_t} + A_{S_t} \left(K^w_{S^t}\right)^{\alpha} \left(L_{S^t}\right)^{1-\alpha} - \int c^y_{S^t}(\kappa,e) dF - \int\int c^o_{z,S^t}(\kappa,e) d\Phi dF \right] d\Pi\left(S^t\right) \\
& + \lambda_{S_0} w_{S_0} \left[L^o_{S_0} + \int \left(1 - h_{S_0}(\kappa,e)\right) dF - L_{S_0} \right] + \lambda_{S_0} r_{S_0} \left[K_{S_0} - K^w_{S_0} - \int \tilde{k}_{S_0}(\kappa,e) dF \right] \\
& + \sum_{t=0}^{\infty} \lambda_{S^t} w_{S^t} \left\{ \int \mathbb{I}_{h_{S^{t-1}}(\kappa,e)=0} dF + \int \left(1 - h_{S^t}(\kappa,e)\right) dF - L_{S^t} \right\} dF d\Pi\left(S^t\right)
\end{aligned}
$$

$$
+ \sum_{t=0}^{\infty} \lambda_{S^t} r_{S^t} \left\{ \int\int \frac{1}{b} \left[\tilde{k}_{S^{t-1}}(\kappa,e)\right]^{\eta} \left(\kappa h_{S^{t-1}}(\kappa,e)\right)^{1-\eta} \max\left(\zeta, 1\text{-}\theta_s\right) z d\Phi dF - K^w_{S^t} - \int \tilde{k}_{S^t}(\kappa,e) dF \right\} d\Pi\left(S^t\right)
$$

$$
+ \sum_{t=0}^{\infty} \lambda_{S^t} \int \mu_{z,S^t}(\kappa,e) \left\{ c^o_{z,S^t}(\kappa,e) - r_{S^t} z (1\text{-}\theta) \frac{1}{b} \left[\tilde{k}_{S^{t-1}}(\kappa,e)\right]^{\eta} \left(\kappa h_{S^{t-1}}(\kappa,e)\right)^{1-\eta} \right\} d\Phi \mathbb{I}_{h_{S^t}(\kappa,e)>0} dF d\Pi\left(S^t\right).
$$

Using the training cost function (16), we can rewrite the above Lagrangian as

$$
\mathcal{L} = \int \gamma_{S_{-1}}(\kappa,e) \int \ln c^o_{z,S_0}(\kappa,e) d\Phi dF + \sum_{t=0}^{\infty} \lambda_{S^t} \left[A_{S_t} \left(K^w_{S^t}\right)^{\alpha} \left(L_{S^t}\right)^{1-\alpha} - r_{S^t} K^w_{S^t} - w_{\tau} L_{S^t} \right]
$$

$$
+ \sum_{t=0}^{\infty} \int \left\{ \begin{array}{c} \gamma_{S^t}(\kappa,e) \left[\ln c^y_{S^t}(\kappa,e) + \beta \int \ln c^o_{z,S^t}(\kappa,e) d\Phi \right] \\ + \lambda_{S^t} [e + w_{S^t} - c^y_{S^t}(\kappa,e) - \varphi_{S^t}\left(k^+_{S^t}(\kappa,e);\kappa\right)] \\ + \lambda_{S^{t+1}} \int [r_{S^{t+1}} \max\left(\zeta, 1-\theta_s\right) z k^+_{S^t}(\kappa,e) - c^o_{z,S^{t+1}}(\kappa,e)] d\Phi \\ + \lambda_{S^{t+1}} \int \mu_{z,S^{t+1}}(\kappa,e) \left[c^o_{z,S^{t+1}}(\kappa,e) - (1-\theta_s) r_{S^{t+1}} z k^+_{S^t}(\kappa,e) \right] d\Phi \end{array} \right\} \mathbb{I}_{h_{\tau}(\kappa,e)>0} dF d\Pi\left(S^{t+1}\right)
$$

$$
+ \sum_{t=0}^{\infty} \int \left\{ \begin{array}{c} \gamma_{S^t}(\kappa,e) \left[\ln c^y_{S^t}(\kappa,e) + \beta \ln c^o_{S^{t+1}}(\kappa,e) \right] \\ + \lambda_{S^t} [e + w_{S^t} - c^y_{S^t}(\kappa,e)] + \lambda_{S^{t+1}} \left[w_{S^{t+1}} - c^o_{S^{t+1}}(\kappa,e) \right] \end{array} \right\} \mathbb{I}_{h_{\tau}(\kappa,e)=0} dF d\Pi\left(S^{t+1}\right),
$$

For those $h_{S^t}(\kappa,e) > 0$, FOC with respect to $c^y_{S^t}(\kappa,e)$ is:

$$
\frac{\gamma_{S^t}(\kappa,e)}{c^y_{S^t}(\kappa,e)} = \lambda_{S^t}
$$

FOC with respect to $c^o_{z,S^{t+1}}(\kappa,e)$:

$$
\frac{\beta \gamma_{S^t}(\kappa,e)}{c^o_{z,S^{t+1}}(\kappa,e)} = \lambda_{S^{t+1}} \left[1 - \mu_{z,S^{t+1}}(\kappa,e)\right] = 0.
$$

Defining $q_{S^{t+1}} = \frac{\lambda_{S^{t+1}}}{\lambda_{S^t}}$, we learn

$$c^o_{z,S^{t+1}}(\kappa,e) = \max\left[z, z^*_{S^{t+1}}(\kappa,e)\right](1-\theta)r_{S^{t+1}}k^+_{S^t}(\kappa,e), \text{ where}$$

$$z^*_{S^{t+1}}(\kappa,e)(1-\theta)r_{S^{t+1}}k^+_{S^t}(\kappa,e) = \frac{\beta}{q_{S^{t+1}}}c^y_{S^t}(\kappa,e).$$

FOC with respect to $k^+_{S^t}(\kappa,e)$:

$$\begin{aligned}\varphi'_{S^t}\left(k^+_{S^t}(\kappa,e);\kappa\right) &= \int q_{S^{t+1}t}r_{S^{t+1}}\int\left[\max\left(\zeta,1-\theta_s\right)z - \mu_{z,S^{t+1}}(\kappa,e)(1-\theta_s)z\right]d\Phi d\Pi\left(S^{t+1}|S^t\right)\\ &= \int r_{S^{t+1}}Q_{S^{t+1}}\left(z^*_{S^{t+1}}(\kappa,e)\right)d\Pi\left(S^{t+1}\right), \text{ where}\end{aligned}$$

$$Q_{S^{t+1}}\left(z^*_{S^{t+1}}(\kappa,e)\right) = q_{S^{t+1}}\left\{\int\max\left(\zeta,1-\theta_s\right)d\Phi_\zeta - (1-\theta_s)\int_{z^*_{S^{t+1}}(\kappa,e)}^\infty\left[z - z^*_t(\kappa,e)\right]d\Phi_z\right\}.$$

We can also choose the Pareto weights $\gamma_{S^t}(\kappa,e)$ to satisfy the budget constraint

$$\begin{aligned}&c^y_{S^t}(\kappa,e) + \varphi_{S^t}\left(k^+_{S^t}(\kappa,e);\kappa\right) + \int c^o_{z,S^{t+1}}(\kappa,e)d\Phi d\Pi\left(S^{t+1}|S^t\right)\\ &= e + w_{S^t} + k^+_{S^t}(\kappa,e)\int q_{S^{t+1}}r_{S^{t+1}}\left[\int\max\left(\zeta,1-\theta_s\right)d\Phi_\zeta\right]d\Pi\left(S^{t+1}|S^t\right)\end{aligned}$$

or

$$(1+\beta)c^y_{S^t} + \varphi_{S^t}\left(k^+_{S^t}(\kappa,e);\kappa\right) = e + w_{S^t} + k^+_{S^t}(\kappa,e)\int r_{S^{t+1}}Q_{S^{t+1}}\left(z^*_{S^{t+1}}(\kappa,e)\right)d\Pi\left(S^{t+1}|S^t\right).$$

These four equations are the same for with the conditions for type (κ,e) trainee in the competitive equilibrium, concerning $c^y_{S^t}(\kappa,e), c^o_{z,S^{t+1}}(\kappa,e)$, $z^*_{S^{t+1}}(\kappa,e)$ and $k^+_{S^t}(\kappa,e)$ for the same $q_{S^{t+1}}, w_{S^t}, r_{S^t}$ and $r_{S^{t+1}}$.

For the first order conditions for $c^y_{S^t}(\kappa,e)$ and $c^o_{S^{t+1}}(\kappa,e)$ for (κ,e) type agents with $h_{S^t}(\kappa,e)=0$, we get

$$c^y_{S^t}(\kappa,e) = \frac{q_{S^{t+1}}}{\beta}c^o_{S^{t+1}}(\kappa,e) = \frac{1}{1+\beta}\left[e + w_{S^t} + \int q_{S^{t+1}}w_{S^{t+1}}d\Pi\left(S^{t+1}|S^t\right)\right].$$

Thus the first order condition for the occupational choice, $I_{h_t(\kappa,e)>0}$, we get $h_t(\kappa,e)>0$ if

$$\begin{aligned}&\left\{\ln c^y_{S^t}(\kappa,e) + \beta\int\int\ln c^o_{z,S^{t+1}}(\kappa,e)d\Phi d\Pi\left(S^{t+1}|S^t\right)\right\}_{h_{S^t}(\kappa,e)>0}\\ &> \left\{\ln c^y_{S^t}(\kappa,e) + \beta\int\ln c^o_{S^{t+1}}(\kappa,e)d\Pi\left(S^{t+1}|S^t\right)\right\}_{h_{S^t}(\kappa,e)>0},\end{aligned}$$

or

$$\begin{aligned}&(1+\beta)\ln\left\{e + w_{S^t} + k^+_{S^t}(\kappa,e)\int r_{S^{t+1}}Q_{S^{t+1}}\left(z^*_{S^{t+1}}(\kappa,e)\right)d\Pi\left(S^{t+1}|S^t\right) - \varphi_{S^t}\left(k^+_{S^t}(\kappa,e);\kappa\right)\right\}\\ &+ \beta\int_{z^*_{S^{t+1}}(\kappa,e)}^\infty[\ln z - \ln z^*_{S^{t+1}}(\kappa,e)]d\Phi_z d\Pi\left(S^{t+1}|S^t\right)\\ &> (1+\beta)\ln\left(e + w_{S^t} + \int q_{S^{t+1}}w_{S^{t+1}}d\Pi\left(S^{t+1}|S^t\right)\right).\end{aligned}$$

This is the same condition with the competitive equilibrium for the occupational choice.

Fro the above all, we find the Pareto weight $\gamma_{S^t}(\kappa, e)$ and $\gamma_{S_{-1}}(\kappa, e)$ with which the solution of social planner's problem corresponds to the competitive equilibrium under the same constraint of the limited commitment.