

Information Acquisition, Efficiency, and Non-Fundamental Volatility

Benjamin Hébert

Stanford and NBER

Jennifer La'O

Columbia and NBER

September 26, 2019

Abstract

This paper analyzes equilibria and efficiency in a class of economies (large games) that feature incomplete information, strategic interaction, and endogenous information acquisition, with beauty contests as a leading example. We adopt the rational inattention approach to endogenous information acquisition but generalize to a large class of information cost structures. Our framework allows agents to learn not only about exogenous states but also about endogenous aggregate actions. We study how the properties of the agents' cost functions are related to the properties of equilibrium and efficient strategies in this class of games. First, we derive conditions under which equilibrium strategies may exhibit non-fundamental volatility; we show how this may be interpreted as a noisy public signal that arises endogenously within a rational inattention framework. Second, we derive a separate set of conditions under which equilibrium information acquisition is efficient. We find that inefficiency in information acquisition occurs when cost functions are such that agents endogenously choose to learn directly about the actions of other agents, as opposed to learning only about exogenous states. Mutual information, the cost function typically used in the rational inattention literature, precludes both non-fundamental volatility and imposes efficiency.

JEL codes: C72, D62, D83

Keywords: Information Acquisition, Large Games, Rational Inattention

1 Introduction

We study a general class of games in which a large number of rationally inattentive agents face uncertainty over exogenous states as well as the endogenous actions of other agents. We allow these rationally inattentive agents to acquire information in an unrestricted way about both the exogenous state of the world and the endogenous actions of others. Within this framework we study the relationship between the properties of the agents' information acquisition cost functions and the properties of the resulting equilibria.

In particular, we ask two questions. First, what properties of the agents' information acquisition costs guarantee that an equilibrium of the game does or does not exhibit non-fundamental volatility? Second, what properties of the agents' information acquisition costs guarantee or preclude the possibility that an equilibrium coincides with the solution to a social planning problem, i.e. is constrained efficient?

Framework and Methodology. We address these questions in the context of a relatively general and oft-studied class of games. In this game we allow there to be one or more types of agents and a continuum of agents within each type. Agents' payoffs are assumed to be functions of their own action, a stochastic payoff-relevant state, and the average (or aggregate) action.

Payoffs depend on the average action: this feature is the strategic interaction in our games. Moreover, agents' actions affect the payoffs of other agents only through the mean (or aggregate) action. This implies that individual agents—each of whom is infinitesimal—need not take into account how their own action affects the aggregate action when making their own strategic choices. This assumption is a defining feature of “large games” and it is analogous to the assumption that agents are “price-takers” in Walrasian markets.

Examples of this class of games have been studied extensively under incomplete information with exogenous information, that is, when agents are endowed with an exogenous signal structure. Examples of this type of large game include abstract beauty contests [[Morris and Shin, 2002](#)], linear-quadratic games of strategic interaction [[Angeletos and Pavan, 2007](#)], New Keynesian nominal price-setting games [[Woodford, 2003](#)], and real business cycle economies with firms making quantity choices, [[Angeletos and La'O, 2010, 2013](#)].

We adopt the rational inattention approach to information acquisition and allow agents to choose signal structures in a relatively unrestricted way. Importantly, we allow agents to learn not only about exogenous states but *also* about endogenous actions, as in [Denti \[2015\]](#).

The ability of rationally-inattentive agents to learn about other agents' actions in this class of games is a novel feature of our paper relative to the previous beauty contest literature.¹ Following the original work of Sims [2003], an extensive literature has studied the implications of rational-inattentive agents. Most of these models can be classified into two strands: (i) models with a single decision maker, optimally choosing his or her information structure, but not participating in a game, or (ii) models of multiple agents playing a game, with all agents are restricted to learn only about exogenous shocks.

The first paper to allow rationally-inattentive agents to learn in an unrestricted way about other agents' actions in a game-theoretic setting is Denti [2015]. We build on his approach but apply it to the class of large games featuring strategic interaction we have just described. Relative to Denti [2015], the “largeness” feature of our class of games permits a simpler definition of equilibrium. Because Denti [2015] considers a finite set of players, agents take into account how their own information choice and actions affect the strategies of other agents, which in turn affects their own learning, etc. For this reason, he considers the long run stationary distribution of a dynamic process of strategic information acquisition. Our definition of equilibrium, which is essentially Nash equilibrium in a static, simultaneous-move game, can also be thought of as the limit of the dynamic process Denti [2015] introduces.

There are two restrictions we place on the agents' ability to acquire information. First, we assume that agents may learn about mean (or aggregate) actions of her own and other types, but we prevent agents from learning about the action of any other *particular* agent. This choice, made for tractability, is again motivated by the “largeness” feature of the game.

Second, we restrict our attention to information costs that are posterior-separable in the terminology of Caplin et al. [2018], meaning that they can be written as the expected value of a divergence between the agent's prior belief and the agent's posterior. A divergence is somewhat analogous to a “distance” from one probability distribution to another. By focusing on the posterior-separable class of information costs, we may characterize properties of information costs in terms of the properties of their associated divergences.

Within this framework, we demonstrate that there is a close connection between certain properties of information costs (divergences) and properties of the equilibria. We find that non-fundamental volatility and constrained efficiency are both connected to the question

¹Outside of the beauty contest literature, learning from prices, which can be thought of as a summary statistic of aggregate actions, is an old idea that goes back at least to e.g. Grossman and Stiglitz [1980]. A recent paper by Angeletos and Sastry [2019] allows rationally inattentive agents to learn from prices in a Walrasian context.

of whether information costs are invariant or monotone with respect to certain classes of transformations of agents’ posteriors. That is, we define various classes of transformations of distributions: these transformations essentially move around the conditional distributions of certain shocks, while leaving the joint distributions of other shocks intact. We then define invariance or monotonicity of divergences with respect to these transformations, and demonstrate that the invariance/monotonicity properties of divergences that we define are critical in determining whether the equilibria of the game exhibit non-fundamental volatility and whether the equilibria are constrained efficient.

Before describing our results, we note that the forms of invariance and monotonicity that we introduce are generalizations of “invariance under compression” (Caplin et al. [2018]) and the concept of invariance described in the literature on information geometry (e.g. Amari and Nagaoka [2007]), applied in related economic applications by Hébert [2018] and Hébert and Woodford [2018a]. The key difference is that all of these papers consider only whether a divergence or cost function is invariant or monotone with respect to all possible transformations, whereas here we show how the answer to the two questions posed in this paper relate to invariance/monotonicity with respect to *specific* classes of transformations.

Payoff-relevant vs. Non-payoff-relevant volatility. Consider our first question: what conditions on information costs guarantee that an equilibrium of the game does or does not exhibit non-payoff-relevant volatility?

We show that if the divergences that define information costs are monotone to transformations of posteriors that rotate the conditional distributions of payoff-irrelevant shocks, keeping the posterior joint distribution of aggregate actions and payoff-relevant states intact, then an equilibrium of the game exists which features zero non-payoff-relevant volatility. The intuition for this result is the following. Agents’ payoffs are functions only of aggregate actions and payoff-relevant shocks (by definition). Thus, agents do not care about payoff-irrelevant shocks per se, only to the extent that they may affect aggregate actions.

This implies that if an agent can learn about payoff-relevant states, payoff-irrelevant states, and actions, she would optimally choose the least costly signal structure that helps her predict aggregate actions and payoff-relevant states. If information costs are monotone with respect to the class of transformations just described, then the least-costly signal structure is also the minimally-informative one: it is the signal structure that allows her to track only aggregate actions and payoff-relevant states but throw away all other conditional information about other objects. If all agents optimally choose to pay no attention to payoff-irrelevant

states conditional on payoff-relevant states and aggregate actions, then the equilibrium fixed point is one in which the aggregate action profile depends solely on payoff-relevant states. Therefore, the equilibrium exhibits zero non-payoff-relevant volatility.

On the other hand, if information costs are instead generically non-monotone in the class of transformations just described, this will not be the case. An agent would still find it optimal to choose the least costly signal structure that allows her to predict aggregate actions and payoff-relevant states. However, in this case, the least costly signal structure is one in which the agent *does not* throw away all information about payoff-irrelevant shocks conditional on payoff-relevant states and aggregate actions. For example, suppose there is a payoff-irrelevant state that is correlated with a payoff-relevant state and it is sometimes cheaper for an agent to look at that “signal” than the payoff-relevant state itself. An agent with these information costs may choose to track the payoff-irrelevant state, i.e. correlate her own action with it, even *conditional* on the aggregate action and payoff-relevant states. In equilibrium, this behavior implies that the mean action exhibits tracks the payoff-irrelevant state, generating non-payoff-relevant volatility. We argue that the payoff-irrelevant state in this context can be interpreted as a noisy public signal.

Constrained Efficiency. Consider now our second question: what conditions on information costs guarantee or preclude the existence of constrained efficient equilibria?

We find that constrained efficiency under endogenous information requires both efficiency of equilibria under exogenous information and efficiency of information acquisition.² The first is a statement about payoffs: we derive conditions on payoffs which ensure efficiency under exogenous information. The second is a statement about information acquisition costs. We show that if cost functions are invariant to transformations of posteriors that instead rotate the conditional distributions of aggregate actions, keeping the posterior joint distribution of payoff-relevant and payoff-irrelevant states intact, then information acquisition is efficient. Both of these properties are generally necessary for equilibria to be efficient under endogenous information acquisition.

To understand this result, consider first the game under exogenous information, in which agents’ strategies are simply mappings from signals to actions. Agents are endowed with an information structure, i.e. a conditional distribution of signals, conditional on exogenous

²A similar result appears in [Colombo et al. \[2014\]](#), but for different reasons. In that paper, the dispersion of agents’ actions affects the welfare of other agents, and acquiring more precise information reduces dispersion. This externality is entirely absent from our model, in which only the mean action affects the welfare of other agents.

shocks (both payoff-relevant and payoff-irrelevant). A constrained planner in this game also cannot choose the information structure but may choose the agents’ strategies in order to maximize welfare.

In the exogenous information game, we derive a sufficient condition on the payoff structure such that the agents’ equilibrium strategies coincide with that chosen by the constrained planner. The condition we derive is a generalization of a condition stated in [Angeletos and Pavan \[2007\]](#). [Angeletos and Pavan \[2007\]](#) consider the class of linear-quadratic games with strategic interactions; they show that efficiency under exogenous information requires both efficiency under *complete* information and an extra requirement on quadratic payoffs that ensure that the “strategic interaction” or degree of coordination faced by agents coincides with that desired by the planner. With these two conditions, [Angeletos and Pavan \[2007\]](#) show that the equilibrium *use* of exogenous information is efficient.

The condition we derive on payoffs applies to a much more general class of payoffs and possible information structures, and provides a different perspective on the source and interpretation of the externalities. In particular, we argue that inefficiency arises from “distributed information externalities,” in which one agent knows that another agent is making a mistake but does not take this into account when choosing her own action.

Having established this, we then assume efficiency under exogenous information and ask what more is needed to ensure efficiency under endogenous information acquisition. That is, does efficiency under exogenous information imply efficiency under endogenous information acquisition when, importantly, agents can learn in an unrestricted way about *both* aggregate states and aggregate actions?³

The answer is no. We show that efficiency in this context requires another form of invariance in the information costs of agents. In particular, we find that if there is efficiency in the use of information and if cost functions are invariant in endogenous variables (aggregate actions), then the equilibrium is constrained efficient. We furthermore show that in the special case of one type of agent, these conditions are also necessary.

A version of the sufficiency result appears in prior work by [Angeletos and Sastry \[2019\]](#), who study constrained efficiency of equilibria in a Walrasian setting. [Angeletos and Sastry \[2019\]](#) consider two cases: one in which agents can track only the exogenous state, and one in which agents can also track the price. They show that the two cases are equivalent if agents’ information costs are invariant in the standard sense (e.g. as in mutual information).

³The inefficiency in information acquisition we identify is distinct from the one in [Colombo et al. \[2014\]](#).

In contrast, we study large games,⁴ and argue that it is a particular kind of invariance that is sufficient and (in a special case) generically necessary for efficiency. This distinction is relevant for cost functions that have been proposed in the literature. For example, while mutual information is the only invariant (in the standard sense) divergence within the class of “Bregman divergences” (Amari and Nagaoka [2007], Caplin et al. [2018]), the Tsallis entropy cost functions (also in this class) proposed by Caplin et al. [2018] are invariant in the sense we show is sufficient for efficiency.

This constrained efficiency result has two implications. First, suppose we had allowed agents to endogenously acquire information, but restricted the set of objects agents can learn from to be *only* the exogenous states—both payoff-relevant and payoff-irrelevant. That is, agents may endogenously learn, but only about exogenous objects. Then, efficiency under exogenous information automatically implies efficiency under information acquisition in this context. Although this result is not well-known in this literature, it has been shown in some specific contexts.⁵

Second, in our more general framework where agents are allowed to learn also about endogenous actions, there is an externality whenever the cost function does not satisfy the invariance condition just described. This inefficiency comes from the following externality: when agents choose to learn about other agents’ actions, they are effectively correlating their actions with other agents’ actions but at the same time not internalizing how this affects the conditional distribution of the aggregate action. If the cost structure is not invariant in the way described, then the equilibrium conditional distribution of the aggregate action affects all agents’ information acquisition costs. That is, agents do not take into account how their actions affect the information acquisition costs of other agents. This is an externality in the eyes of the planner, who would choose a different signal structure than the one that arises in equilibrium.

This externality is familiar because it is similar to the informational externality that arises in any exogenous information framework when agents may observe an exogenous signal about an endogenous object (such as prices).⁶ We derive the particular primitives that give rise to

⁴Focusing on games as opposed to a Walrasian setting sidesteps certain tricky issues. In particular, with non-fundamental volatility and rationally inattentive agents, one must specify how markets clear; this choice can affect the positive implications of the model. Moreover, efficiency in the Walrasian setting requires “complete markets,” including in particular completeness with respect to the realization of agents’ individual signals. If we interpret agents’ signal realizations as perceptual errors (the standard rational inattention interpretation), it is not obvious why they should be contractible.

⁵See e.g. Online Appendix A of Angeletos and La’O [2020]

⁶See e.g. Angeletos and La’O [2008], Angeletos and Pavan [2009], Amador and Weill [2010], Angeletos

this behavior in equilibrium, rather than assume that agents receive a costless signal about endogenous objects.

To summarize, we find that efficiency under exogenous signals and efficiency with endogenous information are equivalent if agents can *only* acquire information about exogenous states of nature. However, if agents can acquire information about the actions of other agents, efficiency requires an additional invariance condition on the form of the agents' information acquisition cost structure.

Further Implications. As stated above, the forms of invariance and monotonicity that we introduce are generalizations of “invariance under compression” [Caplin et al., 2018]. While the literature has focused on divergences that are invariant or monotone with respect to all possible transformations, we classify divergences on whether they are invariant or monotone with respect to specific classes of transformations. The implication of making this distinction is that we can construct examples of cost functions that will lead to neither efficiency nor non-fundamental volatility, both efficiency and non-fundamental volatility, efficiency without non-fundamental volatility, and non-fundamental volatility without efficiency.

For example, the standard cost function in the rational inattention literature, introduced by Sims [2003], is mutual information, and the associated divergence is the Kullback-Leibler divergence. The Kullback-Leibler divergence is invariant in the standard sense and thereby invariant with respect to all classes of transformations we introduce. As a result, within this class of models, if all agents are assumed to have mutual information as their information cost, then equilibria will be efficient and exhibit zero non-fundamental volatility.

Thus, a key message of our paper is that relying on the the mutual information cost function automatically rules out potentially interesting economic behavior. In contrast, several of the alternatives proposed in the literature (the Tsallis entropy costs (Caplin et al. [2018]), the neighborhood-based cost functions of Hébert and Woodford [2018b], and the LLR cost function of Pomatto et al. [2018]) are not invariant with respect to at least one of the classes of transformations we define, and consequently will result in equilibria with inefficiency and/or non-fundamental volatility. These alternatives are motivated in part by experiments (such as Dean and Neligh [2018]) that find that the predictions with the mutual information cost function are unable to match data from single-agent decision problems. Our contribution is to demonstrate that mutual information is not only inconsistent with

et al. [2015]

behavioral evidence, but that using mutual information instead of one of these other proposed alternatives has extremely strong predictions about efficiency and non-fundamental volatility in strategic settings.

Related Literature. [Denti \[2015\]](#) primarily employs the standard mutual information cost function, but allows agents to gather signals about both an exogenous state of nature and the equilibrium actions of other agents, justifying this approach via a stochastic learning model. We adapt the approach of [Denti \[2015\]](#) to games with a continuum of players.⁷

There is an extensive literature on beauty contests that focuses on efficiency and non-fundamental volatility. This literature almost universally employs a linear-quadratic-Gaussian framework. [Angeletos and Pavan \[2007\]](#) characterize efficiency and non-fundamental volatility under dispersed information given exogenous signals. [Hellwig and Veldkamp \[2009\]](#) and [Myatt and Wallace \[2011\]](#) study beauty contests with endogenous information acquisition, and (closest to our paper) [Colombo et al. \[2014\]](#) study efficiency with endogenous information. Our paper departs in several respects from this literature. First, we emphasize a general class of games as opposed to linear-quadratic-Gaussian beauty contests, and we rule out the possibility that the variance of actions within a type affects utilities (a possibility considered by [Angeletos and Pavan \[2007\]](#) and emphasized by [Colombo et al. \[2014\]](#)). Second, because we are not restricted to Gaussian signals, our paper analyzes information acquisition in terms cost functions in a rational inattention problem, and provides results relating the properties of these cost functions to the properties of equilibria. In contrast, [Hellwig and Veldkamp \[2009\]](#), [Myatt and Wallace \[2011\]](#), and [Colombo et al. \[2014\]](#) study endogenous information by allowing agents to choose the precision of Gaussian signals. Third, our approach of adopting the “learning about actions of others” approach of [Denti \[2015\]](#) has no direct analog in this literature that we are aware of; all of the papers mentioned allow agents to acquire information only about exogenous fundamentals.

Allowing agents to learn about the actions of others has many precedents outside of beauty contests and global games. For example, in the classic moral hazard setting of [Holmstrom et al. \[1979\]](#), the principal can receive a signal whose distribution depends on the action of the agent, “monitoring” the agent’s actions. What is novel about our approach (following [Denti \[2015\]](#)) is that we think of agents as monitoring other agents in a simultaneous-move

⁷Other related papers include [Denti \[2017\]](#), who considers network effects and externalities related to information acquisition, and [Rigos et al. \[2018\]](#), who extends the analysis of [Denti \[2015\]](#) under mutual information costs to the case of large coordination games.

game in which agents are ex-ante identical (or can be grouped into a finite number of types). At first, this might seem strange—how can one agent monitor the action of another agent when they must move simultaneously? Note, however, that we could ask the same question of Walrasian equilibrium: how can the agents choose consumption given prices when their consumption determines prices? The answer in both cases is that the static, simultaneous-move game is a stand in for a more dynamic process, in our case the stochastic learning game of [Denti \[2015\]](#).

Layout. This paper is organized as follows. We begin in [Section 2](#) by defining the general class of games we study, and provide as a leading example a simple version of the beauty contest model. In [Section 3](#) we define equilibria under exogenous information and equilibria with endogenous information acquisition and prove existence of both. In [Section 4](#) we characterize the properties of cost functions that determine whether equilibria feature non-fundamental volatility. In [Section 5](#) we define and characterize efficiency under exogenous information, and then describe the properties of cost functions that, combined with efficiency under exogenous information, lead to efficiency with endogenous information acquisition. We conclude in [Section 6](#).

2 The Environment

2.1 Primitives

There are I fundamental “types” of agents, indexed by $i \in \{0, \dots, I - 1\}$. Types determine agents’ payoffs and information acquisition technologies. Within each type, there is a continuum of agents, indexed by $j \in [i, i + 1)$.

Agent j of type i chooses her action, $a^j \in A^i \subseteq \mathbb{R}^L$. For each type $i \in I$, the aggregate action is the average of the actions chosen by agents of that type,

$$\bar{a}^i = \int_i^{i+1} a^j dj.$$

Let $\bar{a} \in \bar{A} \subseteq \mathbb{R}^{L \times I}$ be the vector of aggregate actions for each type.

There is a finite set of fundamental states, $s \in S$. These states, along with aggregate actions \bar{a} , determine the agent’s payoffs. Agents of type i have a utility function $u^i : A^i \times \bar{A} \times S \rightarrow \mathbb{R}$; that is, an agent j of type i who takes action $a^j \in A^i$ in state $s \in S$ when

the aggregate action is $\bar{a} \in \bar{A}$ receives payoff $u^i(a^j, \bar{a}, s)$. We impose the following regularity assumption on the utility functions and action space.

Assumption 1. *For all $i \in I$, A^i is non-empty, convex, and compact, and $u^i(a^j, \bar{a}, s)$ is continuously differentiable on $A^i \times \bar{A}$ for all $s \in S$.*

We impose Assumption 1; it will be sufficient, but not necessary, for our results.⁸ In particular, our results could readily be extended to games with finitely many actions for each type of agent.

The final primitives of the environment are the agents' information acquisition technologies, which we discuss next. Prior to that, we show how this environment nests the typical beauty contest environment.

Example: Beauty Contest Games. Consider the canonical “beauty contest” game. There is only one type of agent, $I = 1$, but within this type there is a continuum of agents indexed by $j \in [0, 1]$. Each agent chooses a one-dimensional action $a^j \in A$ where A is a non-empty, convex, and compact subset of \mathbb{R} . Payoffs are given by

$$u(a^j, \bar{a}, s) = -(1 - \chi)(a^j - \beta(s))^2 - \chi(a^j - \bar{a})^2 \quad (1)$$

where $\bar{a} = \int_0^1 a^j dj$ denotes the mean action, $\beta : S \rightarrow \mathbb{R}$ is an arbitrary function of the fundamental state, and $\chi \in \mathbb{R}$ is a scalar which we assume satisfies $\chi < 1$. This game is nested in our environment and the payoffs in (1) satisfy the conditions in Assumption 1.

The first component of payoffs is a quadratic loss in the distance between the agent's action and the function $\beta(s)$ of the aggregate state; the second component is a quadratic loss in the distance between the agent's action and the mean action. The scalar χ governs the extent of strategic interaction in this game: when $\chi > 0$ actions are said to be “strategic complements,” when $\chi < 0$ actions are said to be “strategic substitutes.” Assumption 1 ensures existence of pure-strategy Nash equilibria of this game under complete information, and $\chi < 1$ to ensures uniqueness. The function β is also the complete information equilibrium strategy. That is, under complete information, the Nash equilibrium strategy profile is given by $a^j(s) = \bar{a}(s) = \beta(s), \forall j$.

The focus of our paper is games with incomplete information and endogenous acquisition of information. We next describe the informational environment.

⁸ Note that Assumption 1 guarantees the existence of mixed strategy Nash equilibria under complete information in games with continuous actions spaces. See, e.g., [Fudenberg and Tirole \[1991\]](#) theorem 1.2

2.2 Shocks and Information Acquisition

Shocks and Priors. In addition to the fundamental payoff-relevant states, $s \in S$, we allow for two types of exogenous non-fundamental states, $r \in R$ and $z \in Z$, each of which are drawn from finite sets. We will interpret these states as being related to a noisy public signal and a sunspot, respectively. We will elaborate on the distinction between these two types of shocks and justify this interpretation in what follows.

Agents are endowed with a common prior $\mu_0(s, r, z)$ over the exogenous states. Let $\mathcal{U}_0 \equiv \Delta(S \times R \times Z)$ denote the space of probability measures over the exogenous states, with $\mu_0 \in \mathcal{U}_0$. Note that the non-fundamental shocks r, z can be independent of the fundamental state, in which case they can be interpreted as pure noise, or correlated, in which case they can be interpreted as noisy signals about the fundamental state. Note that these shocks are non-fundamental in the sense that they do not directly affect payoffs, except to the extent that they affect the aggregate action $\bar{a} \in \bar{A}$.

We define $\bar{a} : S \times R \times Z \rightarrow \bar{A}$ as a function mapping exogenous states to an aggregate action. Let $\bar{\mathcal{A}}$ be the space of all such functions.⁹ One may think of $\bar{a} \in \bar{\mathcal{A}}$ as the “aggregate strategy,” as this function will be determined endogenously by aggregating over the individual agents’ strategies.

We will allow agents to learn not only about the exogenous states, but also about other agents’ actions. Agents will optimally choose which objects to pay attention to; in order to facilitate this choice, we specify their prior over the larger $S \times R \times Z \times \bar{A}$ space. We construct this prior as follows.

Let $\mathcal{V} \equiv \Delta(S \times R \times Z \times \bar{A})$ denote the space of probability measures over $S \times R \times Z \times \bar{A}$. For any prior $\mu_0 \in \mathcal{U}_0$ over exogenous states and any aggregate action function $\bar{a} \in \bar{\mathcal{A}}$, this pair induces a probability measure over the larger space. We let $\phi_{\bar{A}} : \mathcal{U}_0 \times \bar{\mathcal{A}} \rightarrow \mathcal{V}$ denote a mapping from any pair μ_0, \bar{a} to its induced probability measure, defined as follows:

$$\phi_{\bar{A}}\{\mu_0, \bar{a}\}(s, r, z, \bar{a}) = \mu_0(s, r, z) \mathbf{1}(\bar{a} = \bar{a}(s, r, z)), \forall s \in S, r \in R, z \in Z, \bar{a} \in \bar{A} \quad (2)$$

We define the space of all probability measures that may be generated on $S \times R \times Z \times \bar{A}$ by some pair (μ_0, \bar{a}) as

$$\mathcal{V}_0 = \{\nu \in \mathcal{V} : \exists \mu_0 \in \mathcal{U}_0 \text{ and } \bar{a} \in \bar{\mathcal{A}} \text{ s.t. } \nu = \phi_{\bar{A}}\{\mu_0, \bar{a}\}\}.$$

⁹We are restricting the aggregate action to be a deterministic function of (s, r, z) . However, as we will discuss below, the state $z \in Z$ can be thought of as a “sunspot.”

Thus, $\mathcal{V}_0 \subseteq \mathcal{V}$.

Given a prior $\mu_0 \in \mathcal{U}_0$ and an aggregate action function $\bar{\alpha} \in \bar{\mathcal{A}}$, the induced prior on the larger space $\nu_0 \in \mathcal{V}_0$ is given by $\nu_0 = \phi_{\bar{\mathcal{A}}} \{\mu_0, \bar{\alpha}\}$.

Agents' strategies. We now consider the choices of the agents. In the game with exogenous information, an individual agent chooses her own action based on some signal $\omega^j \in \Omega$, where Ω is a signal alphabet with cardinality weakly greater than \mathbb{R}^L (and hence all action spaces). An individual agent's strategy is a mapping from signals to actions, $\alpha^j : \Omega \rightarrow A^i$. Let \mathcal{A}^i be the space of all possible strategies for a player of type i ; in this game we say that an agent chooses an action strategy $\alpha^j \in \mathcal{A}^i$.

When we study games with exogenous information, the action strategies α^j are the only choice variables. In games with endogenous information acquisition, agents also choose the signal alphabet Ω and the structure of the signals ω^j .¹⁰ A signal structure is a conditional probability distribution function

$$\nu^j : S \times R \times Z \times \bar{A} \rightarrow \Delta(\Omega).$$

where $\Delta(\Omega)$ denotes the space of probability measures on Ω ; let \mathcal{V}_Ω be the space of all such functions. That is, $\nu^j(\omega|s, r, z, \bar{a})$ denotes the probability of observing signal $\omega \in \Omega$ conditional on the aggregate state (s, r, z, \bar{a}) . Note that our setup prevents agents from learning about any other *particular* agent's action, but allows agents to learn about the aggregate (or average) actions of her own and other types. In allowing rationally inattentive agents to learn about the equilibrium actions of other agents, our analysis builds on [Denti \[2015\]](#).

To summarize, with endogenous information acquisition, an individual agent chooses both a strategy $\alpha^j \in \mathcal{A}^i$ and a signal structure $\nu^j \in \mathcal{V}_\Omega$ in order to maximize his or her payoffs subject to some cost of information acquisition. We discuss these costs after introducing some notation for signal probabilities and posteriors.

Posterior distributions. Take any signal structure (conditional probability) $\nu^j \in \mathcal{V}_\Omega$ chosen by the agent on the larger space, $S \times R \times Z \times \bar{A}$. A signal structure $\nu^j \in \mathcal{V}_\Omega$ and a prior $\nu_0 \in \mathcal{V}_0$ together induce a joint distribution on $S \times R \times Z \times \bar{A} \times \Omega$. The marginal distribution on

¹⁰A standard result in rational inattention, which will apply in our setting, is that it is without loss of generality to assume the space of signals Ω is the space of actions (A^i for agents of type i).

Ω associated with this joint distribution is the agent's unconditional probability of observing signal $\omega \in \Omega$,

$$\pi \{ \nu^j, \nu_0 \} (\omega) = \sum_{s \in S, r \in R, z \in Z} \int_{\bar{A}} \nu^j (\omega | s, r, z, \bar{a}) \nu_0 (s, r, z, \bar{a}) d\bar{a}, \quad (3)$$

with $\pi \{ \nu^j, \nu_0 \} \in \Delta (\Omega)$.

This joint distribution also induces posteriors (conditional on $\omega \in \Omega$) on both the larger $S \times R \times Z \times \bar{A}$ space and the smaller $S \times R \times Z$ space.

Posteriors on the larger space. We can write the agent's posterior over (s, r, z, \bar{a}) conditional on observing any signal $\omega \in \Omega$ as

$$\nu_\omega \{ \nu^j, \nu_0 \} (s, r, z, \bar{a}) = \frac{\nu^j (\omega | s, r, z, \bar{a}) \nu_0 (s, r, z, \bar{a})}{\pi \{ \nu^j, \nu_0 \} (\omega)}, \quad (4)$$

consistent with Bayes' rule and assuming $\pi \{ \nu^j, \nu_0 \} (\omega) > 0$. Note that, if $\nu_0 \in \mathcal{V}_0$, then $\nu_\omega \in \mathcal{V}_0$ for all $\omega \in \Omega$.

Posteriors on the smaller space. We can write the agent's posterior over (s, r, z) conditional on observing any signal $\omega \in \Omega$ as

$$\mu_\omega \{ \nu^j, \nu_0 \} (s, r, z) = \frac{\int_{\bar{A}} \nu^j (\omega | s, r, z, \bar{a}) \nu_0 (s, r, z, \bar{a}) d\bar{a}}{\pi \{ \nu^j, \nu_0 \} (\omega)}, \quad (5)$$

consistent with Bayes' rule and assuming $\pi \{ \nu^j, \nu_0 \} (\omega) > 0$. Note that $\mu_\omega \in \mathcal{U}_0$ for all $\omega \in \Omega$.

We adopt the convention that, for zero probability signals, the posteriors are equal to the priors over the relevant space.

Costs of information acquisition and posterior-separability. Agents face a cost of information acquisition. We generalize the standard rational inattention setup and define the cost of information acquisition of an agent of type i by a function $C_\Omega^i : \mathcal{V}_\Omega \times \mathcal{V}_0 \rightarrow \mathbb{R}_+$. An agent j of type i which chooses signal structure (conditional distribution) $\nu^j \in \mathcal{V}_\Omega$ given prior ν_0 incurs information costs

$$C_\Omega^i (\nu^j, \nu_0),$$

where the subscript Ω indicates the signal alphabet over which the agent chooses its signal structure. We thereby allow information acquisition costs to vary across types.

Given our discussion above about how to transform signal structures over the larger

space to signal structures over the smaller space, we may at times also use the notation $C_{\Omega}^i(\nu^j, \mu_0; \bar{\alpha})$ to denote the agent’s cost of information acquisition, noting that $\nu_0 = \phi_{\bar{\mathcal{A}}}\{\mu_0, \bar{\alpha}\}$. This notation makes clear that the agent’s cost of choosing a signal structure ν^j might depend on both the exogenous prior μ_0 and (at least potentially) the aggregate action function $\bar{\alpha} \in \bar{\mathcal{A}}$. We discuss this latter possibility shortly.

It is without loss of generality to impose the following assumption on the cost function.

Assumption 2. *For all $\mu_0 \in \mathcal{U}_0$ and $\bar{\alpha} \in \bar{\mathcal{A}}$,*

1. *The cost function $C_{\Omega}^i(\nu^j, \nu_0)$ is zero if the signal structure ν^j is uninformative ($\nu_{\omega}\{\nu^j, \nu_0\} = \nu_0 \forall \omega \in \Omega$ s.t. $\pi^j(\omega) > 0$).*
2. *Take $\nu^j \in \mathcal{V}_{\Omega}$ and $\hat{\nu}^j \in \mathcal{V}_{\hat{\Omega}}$ for some signal alphabets Ω and $\hat{\Omega}$. If ν^j Blackwell-dominates $\hat{\nu}^j$ in the sense of [Blackwell \[1953\]](#), then $C_{\Omega}^i(\nu^j, \nu_0) \geq C_{\hat{\Omega}}^i(\hat{\nu}^j, \nu_0)$.*
3. *The cost function $C_{\Omega}^i(\nu^j, \nu_0)$ is convex in ν^j .*

As discussed by [Caplin and Dean \[2015\]](#), and invoking Lemma 1 of [Hébert and Woodford \[2018a\]](#),¹¹ these assumptions are without loss of generality. The first assumption that uninformative signal structures have zero cost is simply a normalization. Next, any behavior that could be observed for a rationally inattentive agent with a cost function not satisfying these conditions could also be observed for a rationally inattentive agent with a cost function satisfying these conditions. The intuition for this result comes from the possibility of the agent pursuing mixed strategies over actions conditional on a signal realization and over choices of signal structures.

Our next assumption requires that the information costs we study are continuous. This assumption is phrased in a somewhat technical fashion in order to account for the possibility that the signal space Ω is not a finite set.

Observe by the finiteness of $S \times R \times Z$ and $\bar{\mathcal{A}} \subseteq \mathbb{R}^{L \times |I|}$ that $\bar{\mathcal{A}}$ can be viewed as a subset of $\mathbb{R}^{L \times |I| \times |S| \times |R| \times |Z|}$ and endowed with the standard (Euclidean) topology. For the signal structures $\mu^j \in \mathcal{U}_{\Omega}$, we use the topology of weak convergence for each $S \times R \times Z$.

Assumption 3. *Under the topology of weak convergence on \mathcal{V}_{Ω} and \mathcal{V}_0 , the cost functions $C_{\Omega}^i(\nu^j, \nu_0)$ are continuous in the product topology of $\mathcal{V}_{\Omega} \times \mathcal{V}_0$.*

¹¹Lemma 1 of [Hébert and Woodford \[2018a\]](#) allows us to replace the [Caplin and Dean \[2015\]](#) “mixture feasibility” condition with convexity. [Hébert and Woodford \[2018a\]](#) prove it in the context of a finite signal alphabet, but nothing in the proof depends on the alphabet being finite.

Assumption 2 implies continuity in ν^j (due to convexity), holding fixed ν_0 . Assumption 3 adds the requirement of continuity in (ν^j, ν_0) , which is equivalent to continuity in $(\nu^j, \bar{\alpha})$ holding fixed μ_0 .

Next, we restrict attention to information costs that are “posterior-separable,” in the terminology of Caplin et al. [2018]. Posterior-separable cost functions can be written as the expected value of a divergence between the agents’ posterior and prior beliefs. A divergence is a measure of how “close” or “far” two distributions are from one another.¹² To capture the idea that the action of other agents might influence the cost of information, we define these divergences on the larger space of probability measures, \mathcal{V}_0 .

Take any signal structure $\nu^j \in \mathcal{V}_\Omega$ and prior $\nu_0 \in \mathcal{V}_0$. The class of posterior-separable cost functions we study can be written as

$$C_\Omega^i(\nu^j, \nu_0) = \int_\Omega \pi\{\nu^j, \nu_0\}(\omega) D^i(\nu_\omega\{\nu^j, \nu_0\} || \nu_0) d\omega, \quad (6)$$

where $D^i : \mathcal{V}_0 \times \mathcal{V}_0 \rightarrow \mathbb{R}_+$ is a divergence from the agents’ prior ν_0 to posterior ν_ω , convex in its first argument and continuous on $\mathcal{V}_0 \times \mathcal{V}_0$.¹³

Assumption 4. *For all $i \in I$, the cost function C^i is posterior-separable as defined by equation (6).*

The cost function in (6) makes evident why the cost of information acquisition may depend on the aggregate action. Recall that agents are endowed with a common prior $\nu_0 \in \mathcal{V}_0$ over the larger state, given by $\nu_0 = \phi_{\bar{A}}\{\mu_0, \bar{\alpha}\}$. Different aggregate action functions $\bar{\alpha} \in \bar{A}$, induce different priors over the larger state. This affects an individual agent’s cost of information acquisition in two ways. First, the prior enters directly as the second argument of the divergence—intuitively, the function $\bar{\alpha}$ affects the distribution of what the agent is trying to track. Second, the prior enters indirectly through the first argument, that is, the agent’s posterior, ν_ω .

Depending on the shapes of these divergences, different aggregate action functions $\bar{\alpha} \in \bar{A}$ may affect the cost of information acquisition for agents, and may thereby induce agents to

¹²A divergence is a function of two probability measures that is weakly positive and zero if and only if the measures are identical. Unlike a distance, a divergence does not need to be symmetric, and does not necessarily satisfy the triangle inequality.

¹³Convexity in the first argument is implied by Assumption 2 and continuity (under the weak topology) by Assumption 3. Also note that we have defined the divergence D^i on \mathcal{V}_0 rather than the entire space $\mathcal{V} = \Delta(S \times R \times Z \times \bar{A})$; all priors and posteriors in our problem will remain in \mathcal{V}_0 , and therefore it is unnecessary to define the divergence on the larger space.

choose different signal structures in order to minimize these costs.

Invariance. We next describe a concept called “invariance,” and introduce an assumption related to invariance on the divergences that define our posterior-separable cost functions. This assumption gives meaning to our distinction between the payoff-irrelevant states $r \in R$ and $z \in Z$, and explains why we interpret the states $z \in Z$ as sunspots.

Invariant divergences are monotone with respect to “coarsenings” of the state space. They have been described in the information geometry literature (see e.g. [Chentsov \[1982\]](#) or [Amari and Nagaoka \[2007\]](#)), and employed in economics by [Hébert \[2018\]](#) and [Hébert and Woodford \[2018a\]](#). Another term for coarsening is “compression,” and invariant divergences have a close connection to the invariance-under-compression axiom described in [Caplin et al. \[2018\]](#).

The literature has focused on whether or not divergences are monotone with respect to all possible coarsenings of the state space. In contrast, we study divergences that are invariant to some but *not necessarily all* coarsenings of the state space.

We begin by defining a particular type of coarsening that removes information about Z . Let $\mathcal{U}_Z \equiv \Delta(S \times R \times \bar{A})$ denote the space of probability measures on $S \times R \times \bar{A}$ and let $\mu_Z \in \mathcal{U}_Z$ denote a particular distribution on this space; the subscript Z indicates the dimension that is missing, a convention we will follow below. Given any probability distribution $\nu \in \mathcal{V}_0$ on the larger space, we can define the coarsening function $\gamma_Z : \mathcal{V}_0 \rightarrow \mathcal{U}_Z$ by, for all $(s, r, \bar{a}) \in S \times R \times \bar{A}$,

$$\gamma_Z \{\nu\} (s, r, \bar{a}) = \sum_{z \in Z} \nu (s, r, z, \bar{a}). \quad (7)$$

The function γ_Z is a coarsening in the sense that it takes a probability distribution on a larger state space ($S \times R \times Z \times \bar{A}$) and projects it onto a smaller state space ($S \times R \times \bar{A}$). It is essentially “throwing out” all information about z , conditional on (s, r, \bar{a}) ; we again use the subscript Z to indicate that the coarsening γ_Z discards information about z given (s, r, \bar{a}) .

We refer to the opposite transformation, going from the smaller space to the larger space, as an embedding.¹⁴ First, note that there is only one way to “throw out” information about a variable: one simply sums the joint probabilities over that variable. In contrast, there are

¹⁴[Chentsov \[1982\]](#) uses the term “Markov congruent embedding.”

many ways of “gaining” information. We therefore define the set of all possible embeddings.

We say an embedding $\phi_Z : \mathcal{U}_Z \rightarrow \mathcal{V}_0$ is a function that maps probability measures on the smaller $(S \times R \times \bar{A})$ space to measures on the larger $(S \times R \times Z \times \bar{A})$ space; any particular embedding may be defined in terms of its associated conditional distribution $\hat{\phi}_Z(z|s, r, \bar{a})$ as follows:¹⁵

$$\phi_Z \{\mu_Z\} (s, r, z, \bar{a}) = \hat{\phi}_Z(z|s, r, \bar{a}) \mu_Z(s, r, \bar{a}). \quad (8)$$

There are many possible conditional distribution functions $\hat{\phi}_Z$; each maps one-to-one to an embedding operator ϕ_Z . Let Φ_Z be the set of all possible embeddings from \mathcal{U}_Z to \mathcal{V}_0 .

Note that we have in fact already seen an embedding, but one defined on a different dimension of the probability space. The function $\phi_{\bar{A}}\{\mu_0, \bar{\alpha}\}$ defined in (2) is an embedding from a smaller state space $(S \times R \times Z)$ to a larger one $(S \times R \times Z \times \bar{A})$. The associated conditional distribution function for this specific embedding is simply $\hat{\phi}_{\bar{A}}(\bar{a}|s, r, z) = \mathbf{1}(\bar{a} = \bar{\alpha}(s, r, z))$.

Armed with these definitions, we are now in a position to define “invariance with respect to Φ_Z .”

Definition 1. A divergence $D : \mathcal{V}_0 \times \mathcal{V}_0 \rightarrow \mathbb{R}_+$ is invariant with respect to Φ_Z , or **invariant in Z** , if for all $\phi_Z, \phi'_Z \in \Phi_Z$ and $\nu_0, \nu_1 \in \mathcal{V}_0$,

$$D(\nu_1 || \nu_0) \geq D(\phi_Z \{\gamma_Z \{\nu_1\}\} || \phi_Z \{\gamma_Z \{\nu_0\}\}) = D(\phi'_Z \{\gamma_Z \{\nu_1\}\} || \phi'_Z \{\gamma_Z \{\nu_0\}\}). \quad (9)$$

There are two parts to this definition. The first inequality in 9 defines monotonicity in Φ_Z . Intuitively, suppose we “throw away” information about z (i.e. apply γ_Z) from two separate distributions $\nu_0, \nu_1 \in \mathcal{V}_0$, and then add back in information about z in the same way for both distributions (i.e. apply ϕ_Z). Monotonicity requires that making the two distributions more similar in this sense reduces the divergence from one to the other.

The second equality in 9 defines invariance in Φ_Z (as opposed to the weaker property of monotonicity). Suppose we take two distributions $\mu_Z, \mu'_Z \in \mathcal{U}_Z$ on the smaller $S \times R \times \bar{A}$ space and add information about z by applying the same embedding operator to both. Invariance means that it doesn’t matter which embedding operator we use, the resulting distributions will always have the same divergence from one to the other. That is, it does not matter how the coarsened distributions $\mu_Z, \mu'_Z \in \mathcal{U}_Z$ are embedded in the larger space, provided that they are embedded in the *same* way.

¹⁵To ensure that the resulting distribution on $\Delta(S \times R \times Z \times \bar{A})$ remains in \mathcal{V}_0 , we require that for all $s \in S, r \in R$, and $\bar{a}, \bar{a}' \in \bar{A}$, the supports of $\hat{\phi}_Z(\cdot|s, r, \bar{a})$ and $\hat{\phi}_Z(\cdot|s, r, \bar{a}')$ do not intersect.

In other words, throwing away information about z , conditional on (s, r, \bar{a}) , can only decrease costs. We impose this as follows.

Assumption 5. *For all $i \in I$, the divergence D^i associated with the posterior-separable cost function C^i is invariant with respect to Φ_Z .*

This assumption highlights the key distinction between the r and the z states in our model; it is essentially a definition of r relative to z . While neither r nor z are directly payoff-relevant, divergences are invariant with respect to Φ_Z .

The implication of this assumption is that if the agent does not care about z per se, only how it affects \bar{a} , then there is no reason for the agent to acquire any information about z . On the other hand, the agent may find it cheaper to obtain signals correlated with r than to gather no information about r at all, even if she has no particular concern for the value of r . In contrast, conditional on (s, r, \bar{a}) , the agent never has any reason to acquire additional information about z —this would only increase the agent’s information costs.

The following lemma states this idea more formally. Define $\mu_Z^j : S \times R \times \bar{A} \rightarrow \Delta(\Omega)$ as a signal structure (conditional distribution) on the space $S \times R \times \bar{A}$, and let $\mathcal{U}_{Z,\Omega}$ be the set of all such signal structures. That is, $\mu_Z^j(\omega|s, r, \bar{a})$ gives the probability of observing signal $\omega \in \Omega$ conditional on the realization of (s, r, \bar{a}) .

Lemma 1. *Assume that the cost function C_Ω satisfies Assumptions 2, 4, and its associated divergence D satisfies Assumption 5. Fix a prior $\nu_0 \in \mathcal{V}_0$. Take any signal structure $\mu_Z \in \mathcal{U}_{Z,\Omega}$ and define the signal structure $\bar{\nu} \{\mu_Z\} \in \mathcal{V}_\Omega$ by*

$$\bar{\nu} \{\mu_Z\} (\omega|s, r, z, \bar{a}) = \mu_Z (\omega|s, r, \bar{a})$$

for all $s \in S, r \in R, z \in Z, \bar{a} \in \bar{A}, \omega \in \Omega$. Let $\bar{\mathcal{V}}_\Omega \{\mu_Z\}$ be the set of all signal structures $\nu' \in \mathcal{V}_\Omega$ that satisfy the following condition:

$$\mu_Z (\omega|s, r, \bar{a}) = \frac{\sum_{z \in Z} (\nu' (\omega|s, r, z, \bar{a}) \nu_0 (s, r, z, \bar{a}))}{\sum_{z \in Z} \nu_0 (s, r, z, \bar{a})}$$

for all $s \in S, r \in R, \bar{a} \in \bar{A}, \omega \in \Omega$. Note that $\bar{\nu} \{\mu_Z\} \in \bar{\mathcal{V}}_\Omega \{\mu_Z\} \subset \mathcal{V}_\Omega$.

Then, for all priors $\nu_0 \in \mathcal{V}_0$ and all signal structures $\mu_Z \in \mathcal{U}_{Z,\Omega}$,

$$C_\Omega (\bar{\nu} \{\mu_Z\}, \nu_0) \leq C_\Omega (\nu', \nu_0) \quad \forall \nu' \in \bar{\mathcal{V}}_\Omega \{\mu_Z\}.$$

Proof. See the appendix, 7.1. □

The signal structure $\bar{\nu}$ as well as all signal structures $\nu' \in \bar{\mathcal{V}}_{\Omega} \{\mu_Z\}$ are constructed so as to induce the same conditional distribution $\mu_Z(\omega|s, r, \bar{a})$. The only difference between $\bar{\nu}$ and any other ν' is that the latter may generate different conditional probability distributions over $\omega \in \Omega$ given states (s, r, z, \bar{a}) and (s, r, z', \bar{a}) for $z \neq z'$ whereas the former treats these two realizations as the same. This is why we call $\bar{\nu}$ the “minimally informative” out all signal structures that coarsen to μ_Z ; any other signal structure ν' “pays attention to noise” in a way that $\bar{\nu}$ does not.

Lemma 1 states that if the cost function is invariant with respect to Φ_Z , the minimally informative signal structure is also the least costly. In other words, paying attention to z is always costly.

There are two immediate implications of this lemma. The first is that we may write the agent’s problem on the (s, r, \bar{a}) space, with choice variable $\mu_Z \in \mathcal{U}_{Z, \Omega}$. Note that this is possible only because (i) the agent’s utility function does not directly depend on z and (ii) the agent’s costs are invariant with respect to Φ_Z ; if either property did not hold, this simplification would not be valid. The second immediate implication is conditional independence: conditional on (s, r, \bar{a}) , the agent’s signal ω^j will be independent of z .

This behavior is what [Caplin et al. \[2018\]](#) call invariance under compression. Although their paper and the literature in general has focused on divergences that are simply “invariant,” meaning that they are invariant with respect to all possible embeddings, we in contrast have thus far only assumed invariance with respect to one possible embedding, Φ_Z .

2.3 Example: Beauty Contest with endogenous information choice

Consider the simple beauty contest game example described previously. This game has been studied extensively under exogenous information. Here we consider the individual agent’s problem with endogenous information choice.

Individual Agent’s Problem. *Given a prior $\nu_0 \in \mathcal{V}_0$, the agent chooses an action strategy $\alpha^j \in \mathcal{A}^i$ and a signal structure $\nu^j \in \mathcal{V}_{\Omega}$ in order to maximize*

$$\max_{s \in S, r \in R, z \in Z} \sum_{\bar{a}} \int_{\bar{A}} \left(\int_{\Omega} u(\alpha(\omega), \bar{a}, s) \nu^j(\omega|s, r, z, \bar{a}) d\omega \right) \nu_0(s, r, z, \bar{a}) d\bar{a} - C_{\Omega}^i(\nu^j, \nu_0)$$

subject to probability feasibility condition

$$\int_{\Omega} \nu^j(\omega|s, r, z, \bar{a}) d\omega = 1, \quad s \in S, r \in R, z \in Z, \bar{a} \in \bar{A} \quad (10)$$

The agent's first order conditions (assuming interior solutions) are given by

$$\alpha(\omega) = \sum_{s \in S, r \in R, z \in Z} \int_{\bar{A}} [(1 - \chi) \beta(s) + \chi \bar{a}] \nu_{\omega} \{\nu^j, \nu_0\}(s, r, z, \bar{a}) d\bar{a}, \quad \forall \omega \in \Omega, \quad (11)$$

and

$$u(\alpha(\omega), \bar{a}, s) \nu_0(s, r, z, \bar{a}) - \frac{\partial C_{\Omega}^i(\nu^j, \nu_0)}{\partial \nu^j(\omega|s, r, z, \bar{a})} - \kappa(s, r, z, \bar{a}) \nu_0(s, r, z, \bar{a}) = 0 \quad (12)$$

for all $s \in S, r \in R, z \in Z, \bar{a} \in \bar{A}, \omega \in \Omega$ where $\kappa(s, r, z, \bar{a})$ denotes the Lagrange multiplier on the probability feasibility constraint in (10) for state (s, r, z, \bar{a}) .

The FOC in (11) may be rewritten as the familiar best response function encountered in this class of games:

$$\alpha(\omega) = (1 - \chi) \mathbb{E}[\beta(s) | \omega] + \chi \mathbb{E}[\bar{a} | \omega], \quad \forall \omega \in \Omega$$

Thus, given the agent's information structure, the agent's best response is simply to play an action which is a weighted linear combination of her optimal strategy under complete information, $\beta(s)$, and of the average action \bar{a} , conditional on her "signal" ω . The weight $\chi < 1$ dictates how much the agent cares about matching the average action relative to the fundamental.

The less familiar FOC in (12) is the agent's optimality condition for her signal structure. Take any two signals $\omega, \omega' \in \Omega$. Conditional on any realization of the aggregate state, the agent chooses how much probability mass to put on these signals. Rearranging equation (12) for $\omega, \omega' \in \Omega$ conditional on the same (s, r, z, \bar{a}) yields:

$$u(\alpha(\omega), \bar{a}, s) - u(\alpha(\omega'), \bar{a}, s) = \frac{1}{\nu_0(s, r, z, \bar{a})} \left(\frac{\partial C_{\Omega}^i(\nu^j, \nu_0)}{\partial \nu^j(\omega|s, r, z, \bar{a})} - \frac{\partial C_{\Omega}^i(\nu^j, \nu_0)}{\partial \nu^j(\omega'|s, r, z, \bar{a})} \right).$$

Optimality requires that the difference in payoffs across the two signals, given optimal action strategy $\alpha(\omega)$, must be equal to their marginal rate of transformation (in terms of information acquisition costs).

2.4 Examples of Cost Functions

In this sub-section, we provide several examples from the literature of posterior-separable cost functions. The examples we introduce are based on the standard mutual information cost function (Sims [2003]), a cost function based on Tsallis entropy (Caplin et al. [2018]), the neighborhood-based cost function of Hébert and Woodford [2018b], and the LLR cost function of Pomatto et al. [2018]. Each of these examples is posterior-separable and satisfies Assumptions 2, 3, and 4 by construction. We may therefore describe each of these cost functions in terms of their associated divergences $D : \mathcal{V}_0 \times \mathcal{V}_0 \rightarrow \mathbb{R}_+$, and we discuss the implications of invariance with respect to Φ_Z (Assumption 5).

Example 1. Mutual Information

We begin with the standard mutual information cost function. In our context, the associated divergence is the Kullback-Leibler (KL) divergence defined on the space \mathcal{V}_0 ,

$$D_{KL}(\nu_1 || \nu_0) = \sum_{s \in S, r \in R, z \in Z} \int_{\bar{A}} \nu_1(s, r, z, \bar{a}) \ln \left(\frac{\nu_1(s, r, z, \bar{a})}{\nu_0(s, r, z, \bar{a})} \right) d\bar{a},$$

adopting the convention that $0 \ln 0 = 0$. Because the KL divergence is an invariant divergence in the standard (information-geometric) sense, it is invariant with respect to Φ_Z .

Example 2. Tsallis Entropy

The Kullback-Leibler divergence is uniquely defined, within the class of Bregman divergences, by invariance. A Bregman divergence is defined in our context using a convex function $H : \mathcal{V}_0 \rightarrow \mathbb{R}$,

$$D_H(\nu_1 || \nu_0) = H(\nu_1) - H(\nu_0) - (\nu_1 - \nu_0) \cdot \nabla H(\nu_0),$$

where $\nabla H(\cdot)$ denotes the gradient. The particular H function that defines the Kullback-Leibler divergence is Shannon’s entropy. Caplin et al. [2018] call cost functions based on Bregman divergences “uniformly posterior-separable.”

Caplin et al. [2018] criticize the mutual information cost function by demonstrating that the behavioral implications of invariance are inconsistent with experimental evidence. They propose as an alternative Bregman divergences based on Tsallis entropy. Tsallis entropy is a generalization of Shannon’s entropy. In our context, to ensure invariance with respect to

Φ_Z , we treat the Z dimension differently than the other dimensions. In our context, the H function describing Tsallis entropy is defined, using the parameter $\rho \in \mathbb{R}$,

$$H_{TS,\rho}(\nu) = \frac{1}{1-\rho} \left(1 - \sum_{s \in S, r \in R} \int_{\bar{A}} \left[\sum_{z \in Z} \nu(s, r, z, \bar{a}) \right]^\rho d\bar{a} \right).$$

In the limit as $\rho \rightarrow 1$, this function approaches (the negative of) Shannon's entropy.

Observe that this H function is really defined on the \mathcal{U}_Z space, since it in effect applies the γ_Z coarsening to its argument. An implication of this assumption is that it is costless for an agent with the Tsallis-based cost function to observe $z \in Z$. In contrast, in the previous (mutual information) example, observing $z \in Z$ is costly. However, in both cases, invariance with respect to Φ_Z is satisfied.

Away from the $\rho \rightarrow 1$ limit, the Bregman divergence based on Tsallis entropy, $D_{TS,\rho}$, is not invariant. However, it is still invariant to certain transformations, even though it is not invariant in the standard sense. We will discuss the particular invariance properties of Tsallis entropy and the resulting equilibrium implications as part of our analysis below.

Example 3. Neighborhood-Based Cost Functions

In contrast, the neighborhood-based cost functions of Hébert and Woodford [2018b] are not in general invariant in any sense. These cost functions are also Bregman divergences, but defined with a different H function. To define a neighborhood-based cost function, we need to define a topology on the space $S \times R \times Z \times \bar{A}$. A straightforward example is if each of these spaces is a subset of the real numbers, in which case a possible topology is the standard Euclidean topology.

A topology defines a set of neighborhoods, \mathbf{X} , whose elements are sets of points in $S \times R \times Z \times \bar{A}$. Given a neighborhood $x \in X$, we can define the probability of being in that neighborhood under measure ν as

$$\nu(x) = \sum_{s \in S, r \in R, z \in Z} \int_{\bar{A}} \nu(s, r, z, \bar{a}) \mathbf{1}\{(s, r, z, \bar{a}) \in x\} d\bar{a}.$$

The H function associated with the neighborhood-based cost function is

$$H_N(\nu) = \sum_{x \in X} c_x \sum_{s \in S, r \in R, z \in Z} \int_{\bar{A}} \nu(s, r, z, \bar{a}) \ln \left(\frac{\nu(s, r, z, \bar{a})}{\nu(x)} \right) \mathbf{1}\{(s, r, z, \bar{a}) \in x\} d\bar{a},$$

where c_x is a weakly positive constant specific to each neighborhood.¹⁶

To ensure that invariance with respect to Φ_Z is satisfied, we will assume that if (s, r, z, \bar{a}) is in some neighborhood x with $c_x > 0$, then (s, r, z', \bar{a}) is also in x , for all $x \in \mathbf{X}$ such that $c_x > 0$, $s \in S$, $r \in R$, $\bar{a} \in \bar{A}$, and $z \in Z$. Under this assumption, the topology in fact imposes no structure on Z , and invariance with respect to Φ_Z follows from the invariance properties of the KL divergence.

Example 4. LLR Cost Functions

Our previous two examples were based on Bregman divergences. Pomatto et al. [2018] propose, based on axiomatic considerations, a non-Bregman divergence and call the associated cost functions the “LLR” cost functions. From their axioms, Pomatto et al. [2018] derive a distance-like function between any two points in the state space.

Their divergence is, adapted to our context and assuming mutually absolutely continuous ν_1 and ν_0 ,

$$D_{LLR}(\nu_1 || \nu_0) = \sum_{s \in S, r \in R} \sum_{s' \in S, r' \in R} \int_{\bar{a} \in \bar{A}'} \int_{\bar{a}' \in \bar{A}'} \beta(s, r, \bar{a}, s', r', \bar{a}') F(s, r, \bar{a}, s', r', \bar{a}') d\bar{a}' d\bar{a}$$

where

$$F(s, r, \bar{a}, s', r', \bar{a}') = \frac{\sum_{z \in Z} \nu_1(s, r, z, \bar{a})}{\sum_{z \in Z} \nu_0(s, r, z, \bar{a})} \ln \left(\frac{\sum_{z \in Z} \nu_1(s, r, z, \bar{a})}{\sum_{z \in Z} \nu_1(s', r', z, \bar{a}')} \right) - \ln \left(\frac{\sum_{z \in Z} \nu_0(s, r, z, \bar{a})}{\sum_{z \in Z} \nu_0(s', r', z, \bar{a}')} \right),$$

$\beta(\cdot)$ is a distance-like function between points in the state space, and

$$\bar{A}' = \left\{ \bar{a}' \in \bar{A} \mid \sum_{z \in Z} \nu_0(s', r', z, \bar{a}') > 0 \right\}.$$

Note that, as in the Tsallis case, we have made it costless for the agent to observe $z \in Z$, ensuring that invariance with respect to Φ_Z is satisfied.

Like the neighborhood-based cost function, the LLR cost function induces a notion of distance on the state space. We will find, as a result, that the two cost functions make similar predictions with respect to non-fundamental volatility and the efficiency of equilibria.

Having laid out our assumptions on payoffs and information costs, and provided examples of each, we next define equilibrium and prove its existence.

¹⁶This example uses Shannon’s entropy within each neighborhood; Hébert and Woodford [2018b] define a more general class using a generalized version of Shannon’s entropy.

3 Equilibrium Definition and Equilibrium Existence

In this section we begin by defining our equilibrium concept under two scenarios: (i) the game with exogenous information and (ii) the game with endogenous information acquisition. We then establish equilibrium existence under both scenarios.

3.1 Equilibrium Definitions.

With exogenous information, the class of games we study may be thought of as large games with dispersed information in which agents only choose their action strategy. Adding endogenous information changes the focus of our analysis from the agent's choice of actions given their signals to their choice of signals.

Consider first the game under the assumption of exogenous information. We study signal structures are exogenous in two senses: they are fixed and not chosen by the agents, and the distributions of signals do not depend on $\bar{a} \in \bar{A}$, and endogenous object. Let $\mathcal{V}_{\bar{A}, \Omega} \subset \mathcal{V}_{\Omega}$ be the set of signal structures $\nu^j : S \times R \times Z \times \bar{A} \rightarrow \Delta(\Omega)$ such that, for all $(\omega, s, r, z) \in \Omega \times S \times R \times Z$ and $\bar{a}, \bar{a}' \in \bar{A}$,

$$\nu^j(\omega|s, r, z, \bar{a}) = \nu^j(\omega|s, r, z, \bar{a}').$$

As discussed in the introduction, these kinds of exogenous signal structures have been the focus of the literature on beauty contests.

Anticipating the endogenous information game, we restrict signal distributions to be identical within type. We also assume that, conditional on (s, r, z) , the realizations of signals within and across types are independent. That is, it is the distributions, not the realizations, that are identical within a type.

Each agent is infinitesimal, meaning that the agent treats the joint distribution of payoff relevant and irrelevant states and aggregate actions as exogenous. We will focus on “type-symmetric Bayesian Nash equilibria” in which each agent within a type chooses the same action strategy. We will also apply the law of large numbers and require in our equilibrium definition that each agent's average action a^j be consistent with the average action for her type, \bar{a}^i .

We will describe the problem of the agent under exogenous information in a slightly unusual way to emphasize the connection between the problem with exogenous information and the problem with endogenous information. Mixed action strategies are mappings from

the realizations of the signals Ω to distributions over actions, $\Delta(A^i)$. Given any mixed strategy $\sigma^i : \Omega \rightarrow \Delta(A^i)$ and exogenous signal structure $\nu_\Omega^i \in \mathcal{V}_\Omega$, we can define the induced conditional distribution over actions, $\nu_A^i : S \times R \times Z \times \bar{A} \rightarrow \Delta(A^i)$,

$$\nu_A^i(a|s, r, z, \bar{a}) = \int_\Omega \sigma^i(a|\omega) \nu_\Omega^i(\omega|s, r, z, \bar{a}) d\omega. \quad (13)$$

Let \mathcal{V}_A^i be the set of all conditional distribution over actions. Note that, because $\nu_\Omega^i \in \mathcal{V}_{\bar{A}, \Omega}$, the conditional distribution ν_A^i does not in fact depend on \bar{a} .

Observe that the mapping σ can be thought of as a “garbling” in the sense of [Blackwell \[1953\]](#); that is, by adding noise in the relationship between signals and actions, the mixed strategies σ ensure that the distributions ν_A^i are weakly Blackwell-dominated by the distributions ν_Ω^i . In fact, by Blackwell’s theorem, the set of conditional action distributions $\nu_A^i \in \mathcal{V}_A^i$ that can be feasibly created by any mixed strategy σ are precisely those that are Blackwell-dominated by ν_Ω^i . Let $\mathcal{V}_{BD}^i(\nu_\Omega^i) \subseteq \mathcal{V}_A^i$ be the subset of \mathcal{V}_A^i Blackwell-dominated by ν_Ω^i , noting again that these conditional distributions are not in fact a function of \bar{a} .

Our definition of equilibrium under exogenous information treats $\nu_A^i \in \mathcal{V}_{BD}^i(\nu_\Omega^i)$ as the choice variable of the agent.

Definition 2. Given a prior $\mu_0 \in \mathcal{U}_0$ and a set of signal distributions $\{\nu_\Omega^i \in \mathcal{V}_{\bar{A}, \Omega}^i\}_{i \in I}$, a **type-symmetric Bayesian Nash equilibrium (TSBNE)** of the game under exogenous information is a set of strategies $\{\nu_A^i \in \mathcal{V}_A^i\}_{i \in I}$ and an aggregate action profile $\bar{\alpha} \in \bar{\mathcal{A}}$ such that

- (i) For each $i \in I$, the strategies $\nu_A^i \in \mathcal{V}_{BD}^i(\nu_\Omega^i)$ are best responses:

$$\nu_A^i \in \sup_{\nu_A^i \in \mathcal{V}_{BD}^i(\nu_\Omega^i)} \sum_{s \in S, r \in R, z \in Z} \int_{\bar{A}} \left[\int_{A^i} u^i(a^j, \bar{a}, s) \nu_A^j(a^j|s, r, z, \bar{a}) da^j \right] \nu_0\{\mu_0, \bar{\alpha}\}(s, r, z, \bar{a}) d\bar{a}$$

- (ii) for all $i \in I$, the mean is consistent with the type’s average action

$$\int_{A^i} a^j \nu_A^i(a^j|s, r, z, \bar{\alpha}(s, r, z)) da^j = \bar{\alpha}^i(s, r, z) \quad \forall i \in I, s \in S, r \in R, z \in Z, s.t. \mu_0(s, r, z) > 0.$$

Viewed from this perspective, the game under exogenous information is not very different from the game under endogenous information. In fact, the definition is almost identical, except that instead of restricting strategies to be Blackwell-dominated by ν_Ω^i , we will allow any strategies in \mathcal{V}_A^i , subject to the convex cost of information $C_A^i(\cdot)$. Put another way,

we are replacing a restriction on a convex subset of \mathcal{V}_A^i (i.e. $\mathcal{V}_{BD}^i(\nu_\Omega^i)$) with a convex cost function on \mathcal{V}_A^i . Our definition with endogenous information follows.

Definition 3. Given a common prior $\mu_0 \in \mathcal{U}_0$, a **type-symmetric Bayesian Nash equilibrium (TSBNE)** of the game under endogenous information is a set of strategies $\{\nu_A^i \in \mathcal{V}_A^i\}_{i \in I}$ and an aggregate action profile $\bar{\alpha} \in \bar{\mathcal{A}}$ such that

(i) For each $i \in I$, the strategies $\nu_A^i \in \mathcal{V}_A^i$ are best responses,

$$\nu_A^i \in \sup_{\nu_A^j \in \mathcal{V}_A^j} \sum_{s \in S, r \in R, z \in Z} \int_{\bar{A}} \left[\int_{A^i} u^i(a^j, \bar{a}, s) \nu_A^j(a^j | s, r, z, \bar{a}) da^j \right] \nu_0\{\mu_0, \bar{\alpha}\}(s, r, z, \bar{a}) d\bar{a} \quad (14)$$

$$- C_A^i(\nu_A^j, \nu_0\{\mu_0, \bar{\alpha}\}),$$

(ii) for all $i \in I$, the mean is consistent with the type's average action,

$$\int_{A^i} a^j \nu_A^i(a^j | s, r, z, \bar{\alpha}(s, r, z)) da^j = \bar{\alpha}^i(s, r, z) \quad \forall i \in I, s \in S, r \in R, z \in Z, s.t. \mu_0(s, r, z) > 0. \quad (15)$$

To streamline our exposition, we have invoked the usual result in rational inattention problems that it is without loss of generality to equate signals and actions. That is, instead of writing the problem as being over choice variables $\sigma^j : \Omega \rightarrow \Delta(A^i)$ and $\nu^j \in \mathcal{V}_\Omega^i$, we have written the problem as a choice over the conditional distribution of actions, $\nu_A^i \in \mathcal{V}_A^i$.

In the problem with exogenous information, we have assumed that the signals are not a function of the endogenous actions \bar{a} . In the problem with endogenous information, the following lemma demonstrates that it is without loss of generality to make this assumption.

Lemma 2. *Under Assumptions 2, 3, and 4, in the equilibrium with endogenous information acquisition (Definition 3), it is without loss of generality to assume that the signals ν_A^i do not depend on \bar{a} conditional on (s, r, z) ; that is, $\nu_A^i \in \mathcal{V}_{\bar{A}, A}^i$ for all $i \in I$.*

Proof. See the appendix, 7.2. □

The intuition for this lemma is straight-forward: zero-probability (s, r, z, \bar{a}) values have no impact on either unconditional signal probabilities or posteriors, and therefore do not change cost functions or expected utility.

3.2 Equilibrium Existence

Our first result shows that a TSBNE exist under both exogenous and endogenous information. The result uses Kakutani’s fixed point theorem in the usual way, relying on the finiteness of $S \times R \times Z$, the continuity of the utility function, and (in the endogenous information case) the convexity and continuity of the information cost function. We have combined the results for the endogenous and exogenous information cases to emphasize their similarities.

Proposition 1. *Under Assumption 1, a TSBNE of the game under exogenous information (Definition 2) exists. Under Assumptions 1, 2, 3, and 4, a TSBNE of the game under endogenous information (Definition 3) exists.*

Proof. See the appendix, 7.3. □

We have established that our equilibria exist, and can now begin to study these equilibria. We will begin by investigating under what circumstances our equilibrium do and do not exhibit “non-payoff-relevant” volatility. We will then investigate the circumstances under which the equilibria are constrained efficient, defining constrained efficiency as being identical to the solution of a planner’s problem. For both of these results, the focus of our investigation will be the relationship between properties of the information costs C^i , and in particular the associated divergences D^i , and the properties of the equilibrium with endogenous information acquisition.

4 Non-Payoff-Relevant Volatility in Equilibrium

We have emphasized that the exogenous states $r \in R$ and $z \in Z$ are not directly relevant to payoffs, and interpreted them as a common signal and a sunspot, respectively. We next consider the question of whether the equilibrium aggregate actions depend on these variables. We begin by defining measurability in the context of the aggregate action function. We will say the aggregate action function $\bar{\alpha}$ is (r,s)-measurable if it does not depend on z , and s-measurable if it does not depend on r nor z .

Definition 4. A TSBNE is **(r,s)-measurable** if $\bar{\alpha}(s, r, z) = \bar{\alpha}(s, r, z')$ for all $s \in S$, $r \in R$, and $z, z' \in Z$. A TSBNE is **s-measurable** if $\bar{\alpha}(s, r, z) = \bar{\alpha}(s, r', z')$ for all $s \in S$, $r, r' \in R$, $z, z' \in Z$.

In this section we find sufficient conditions under which (r,s)-measurable and s-measurable equilibria exist.

4.1 Existence of (r,s)-measurable equilibria.

Our first result demonstrates that an (r,s)-measurable equilibrium always exists. This result is an implication of our focus on large games in which individual agents take the aggregate action function as given, the continuity of utility functions, and (critically) our assumption of invariance on Φ_Z (Assumption 5).

Proposition 2. *Assume that payoffs $\{u^i\}_{i \in I}$ satisfy Assumption 1, that cost functions $\{C_\Omega^i\}_{i \in I}$ satisfy Assumptions 2, 3, and 4, and 5. Then an (r,s)-measurable TSBNE of the game under endogenous information (Definition 3) exists.*

Proof. See the appendix, 7.4. □

Our proof of this result is essentially a restatement of our existence proof combined with an application of Lemma 1, our lemma characterizing behavior under Φ_Z invariance. The key observation is that, with Φ_Z invariance, agents optimally choose actions that conditional on (s, r, \bar{a}) , are independent of z . As a result, if agents expect an (r,s)-measurable \bar{a} function, they will best-respond with a policy whose mean action is indeed (r,s)-measurable.

We have suggested interpreting the states $z \in Z$ as a sunspot. What Proposition 2 demonstrates is that sunspots are not required for the existence of equilibrium in our setting. Note that we do not mean to imply the non-existence of equilibria with sunspots, only that equilibria without sunspots exist. This is a common result in large games, but would not always be expected in games with a finite number of players.

4.2 Existence of s-measurable equilibria.

We next consider sufficient conditions under which an s-measurable equilibrium exists. Note that s-measurable equilibria are the equilibria which contain zero non-fundamental volatility, explaining why we are interested in the existence or non-existence of such equilibria.

We continue to assume divergences satisfy invariance with respect to Z , but we now introduce an additional form of invariance which we refer to as RZ-monotonicity. Just as Z -invariance led to (r,s)-measurability, we now show that RZ-monotonicity leads to s-measurability. The intuition is essentially identical to our previous result: with RZ-monotonicity, best responses will be independent of (r, z) conditional on (s, \bar{a}) and which will imply that best-responses to an s-measurable \bar{a} function will be s-measurable.

We begin by defining a coarsening with respect to both R and Z . Let $\mathcal{U}_{RZ} \equiv \Delta(S \times \bar{A})$ denote the space of probability measures on $S \times \bar{A}$ and let $\mu_{RZ} \in \mathcal{U}_{RZ}$ denote a particular

distribution on this space; again the subscript RZ indicates the dimensions that are missing. Given any probability distribution $\nu \in \mathcal{V}_0$ on the larger space, we define the coarsening function $\gamma_{RZ} : \mathcal{V}_0 \rightarrow \mathcal{U}_{RZ}$ by

$$\gamma_{RZ} \{\nu\} (s, \bar{a}) = \sum_{r \in R, z \in Z} \nu (s, r, z, \bar{a}). \quad (16)$$

This function thereby coarsens in both r and z : it “throws away” all information about r and z conditional on (s, \bar{a}) .

We similarly define an embedding $\phi_{RZ} : \mathcal{U}_{RZ} \rightarrow \mathcal{V}_0$ as a function that maps probability measures on the smaller $(S \times \bar{A})$ space to measures on the larger $(S \times R \times Z \times \bar{A})$ space; any particular embedding may be defined in terms of its associated conditional distribution $\hat{\phi}_{RZ} (r, z | s, \bar{a})$ as follows:¹⁷

$$\phi_{RZ} \{\mu_{RZ}\} (s, r, z, \bar{a}) = \hat{\phi}_{RZ} (r, z | s, \bar{a}) \mu_{RZ} (s, \bar{a}). \quad (17)$$

Again there are many possible conditional distribution functions $\hat{\phi}_{RZ}$; each maps one-to-one to a particular embedding operator ϕ_{RZ} . Let Φ_{RZ} be the set of all possible embeddings from \mathcal{U}_{RZ} to \mathcal{V}_0 .

We next define a composition of the coarsening operation and a *specific* embedding $\phi_{RZ} \in \Phi_{RZ}$. We let $\eta_{RZ} : \mathcal{V}_0 \times \mathcal{V}_0 \rightarrow \mathcal{V}_0$ denote the operation that coarsens its first argument in both r and z , then embeds using the conditional distribution of its second argument. That is,

$$\eta_{RZ} \{\nu_1, \nu_0\} (s, r, z, \bar{a}) = \begin{cases} \frac{\nu_0(s, r, z, \bar{a})}{\gamma_{RZ}\{\nu_0\}(s, \bar{a})} \gamma_{RZ} \{\nu_1\} (s, \bar{a}) & \text{if } \gamma_{RZ} \{\nu_0\} (s, \bar{a}) > 0 \\ 0 & \text{if } \gamma_{RZ} \{\nu_0\} (s, \bar{a}) = 0. \end{cases}$$

To apply this operation, we require that $\gamma_{RZ} \{\nu_1\} (s, \bar{a})$ be absolutely continuous with respect to $\gamma_{RZ} \{\nu_0\} (s, \bar{a})$. Intuitively, this operation takes the distribution $\nu_1 \in \mathcal{V}_0$, discards its conditional distribution of (r, z) given (s, \bar{a}) , but then replaces it with the conditional distribution from ν_0 , that is, embeds using $\hat{\phi}_{RZ} (r, z | s, \bar{a}) = \frac{\nu_0(s, r, z, \bar{a})}{\gamma_{RZ}\{\nu_0\}(s, \bar{a})}$. The end result is a distribution that is “more like” ν_0 than ν_1 was originally. By construction, $\eta_{RZ} \{\nu_0, \nu_0\} = \nu_0$.

Armed with this composition of operators, we define RZ-monotonicity as follows.

Definition 5. A divergence $D : \mathcal{V}_0 \times \mathcal{V}_0 \rightarrow \mathbb{R}_+$ is **RZ-monotone** if for all $\nu_0, \nu_1 \in \mathcal{V}_0$ such

¹⁷Again to ensure that the resulting distribution on $\Delta(S \times R \times Z \times \bar{A})$ remains on \mathcal{V}_0 , for all $s \in S$ and $\bar{a}, \bar{a}' \in \bar{A}$, we require that the supports of $\hat{\phi}_{RZ}(\cdot | s, z, \bar{a})$ and $\hat{\phi}_{RZ}(\cdot | s, z, \bar{a}')$ do not intersect.

that $\gamma_{RZ} \{\nu_1\} \ll \gamma_{RZ} \{\nu_0\}$,

$$D(\nu_1 || \nu_0) \geq D(\eta_{RZ} \{\nu_1, \nu_0\} || \nu_0). \quad (18)$$

where $\eta_{RZ} \{\nu_0, \nu_0\} = \nu_0$.

This form of monotonicity captures the idea that if we make ν_1 more like ν_0 in the sense just described, this should reduce the divergence from ν_0 to ν_1 . While this property seems rather intuitive, we will show that it has strong (and perhaps undesirable) implications for behavior.

Monotonicity in RZ is a weaker property than invariance in RZ . Recall our definition of invariance in Φ_Z in Definition 1 and suppose we were to also assume invariance in Φ_R . The first inequality in (18) looks similar to the inequality in (9). However, in (18) we use a *particular* embedding: that associated with the conditional distribution of ν_0 , $\hat{\phi}_{RZ}(r, z | s, \bar{a}) = \frac{\nu_0(s, r, z, \bar{a})}{\gamma_{RZ}\{\nu_0\}(s, \bar{a})}$. As already noted, this embedding ensures that if one were to apply the composition operator η_{RZ} to ν_0 it would get itself back: $\eta_{RZ} \{\nu_0, \nu_0\} = \nu_0$.

On the other hand RZ-invariance, or invariance with respect to both Φ_Z and Φ_R , would further require that after coarsening in both r and z one could then embed using *any* conditional distribution and the resulting “distance” would be equal. Formally, RZ-invariance would require that, for all $\nu_2 \in \mathcal{V}_0$ satisfying the required absolute continuity,

$$D(\eta_{RZ} \{\nu_1, \nu_0\} || \eta_{RZ} \{\nu_0, \nu_0\}) = D(\eta_{RZ} \{\nu_1, \nu_2\} || \eta_{RZ} \{\nu_0, \nu_2\}).$$

That is, RZ-invariance is RZ-monotonicity plus the additional requirement that the conditional distribution of (r, z) conditional on (s, \bar{a}) under ν_0 is irrelevant. Consequently, RZ-invariance implies RZ-monotonicity, but the reverse is not true.

Armed with this definition, we next present the analog of Lemma 1, but with the additional assumption of RZ-monotonicity. We let $\mu_{RZ}^j : S \times \bar{A} \rightarrow \Delta(\Omega)$ denote a signal structure (conditional distribution) on the space $S \times \bar{A}$ and let $\mathcal{U}_{RZ, \Omega}$ be the set of all such signal structures. That is, $\mu_{RZ}^j(\omega | s, \bar{a})$ gives the probability of observing signal $\omega \in \Omega$ conditional on the realization of (s, \bar{a}) .

Lemma 3. *Assume that the cost function C_Ω satisfies Assumptions 2, 4, and 5. Fix a prior $\nu_0 \in \mathcal{V}_0$. Take any signal structure $\mu_{RZ} \in \mathcal{U}_{RZ, \Omega}$ and define the minimally-informative signal*

structure $\bar{\nu} \{\mu_{RZ}\} \in \mathcal{V}_\Omega$ by

$$\bar{\nu} \{\mu_{RZ}\} (\omega|s, r, z, \bar{a}) = \mu_{RZ} (\omega|s, \bar{a})$$

for all $s \in S, r \in R, z \in Z, \bar{a} \in \bar{A}, \omega \in \Omega$. Let $\mathcal{V}'_\Omega \{\mu_{RZ}\}$ be the set of all signal structures $\nu' \in \mathcal{V}_\Omega$ that satisfy the following condition:

$$\mu_{RZ} (\omega|s, \bar{a}) = \frac{\sum_{r \in R, z \in Z} (\nu' (\omega|s, r, z, \bar{a}) \nu_0 (s, r, z, \bar{a}))}{\sum_{r \in R, z \in Z} \nu_0 (s, r, z, \bar{a})}$$

for all $s \in S, r \in R, \bar{a} \in \bar{A}, \omega \in \Omega$. Note that $\bar{\nu} \{\mu_{RZ}\} \in \mathcal{V}'_\Omega \{\mu_{RZ}\} \subset \mathcal{V}_\Omega$.

(i) If the divergence D associated with the cost function is RZ -monotone, then for all priors $\nu_0 \in \mathcal{V}_0$ and all signal structures $\mu_{RZ} \in \mathcal{U}_{RZ, \Omega}$,

$$C_\Omega (\bar{\nu} \{\mu_{RZ}\}, \nu_0) \leq C_\Omega (\nu', \nu_0) \quad \forall \nu' \in \mathcal{V}'_\Omega \{\mu_{RZ}\}.$$

(ii) If the divergence D associated with the cost function is differentiable with respect to ν_1 at $\nu_1 = \nu_0$, and for all priors $\nu_0 \in \mathcal{V}_0$ and all signal structures $\mu_{RZ} \in \mathcal{U}_{RZ, \Omega}$

$$C_\Omega (\bar{\nu} \{\mu_{RZ}\}, \nu_0) \leq C_\Omega (\nu', \nu_0) \quad \forall \nu' \in \mathcal{V}'_\Omega \{\mu_{RZ}\},$$

then D is RZ -monotone.

Proof. See the appendix, 7.5. □

As in Lemma 1, all signal structures $\nu' \in \mathcal{V}'_\Omega \{\mu_{RZ}\}$, including $\bar{\nu}$, are constructed so as to induce the same conditional distribution $\mu_{RZ} (\omega|s, \bar{a})$. The signal structure $\bar{\nu}$ is the “minimally informative” one in the sense that its conditional distributions of signals depend on neither r nor z .

Lemma 3 states that RZ -monotonicity is equivalent to the statement that the minimally informative signal structure $\bar{\nu} \{\mu_{RZ}\}$ is the least-costly of all signal structures that coarsen to μ_{RZ} . The distinction between monotonicity and invariance is what allows us to prove an if-and-only-if result. Note that the additional assumption of differentiability at $\nu_1 = \nu_0$ is imposed in the “only-if” part of the lemma but not in the “if” part of the lemma.

Armed with this result, we demonstrate that RZ -monotonicity is sufficient for the existence of s -measurable equilibria.

Proposition 3. *Assume that payoffs $\{u^i\}_{i \in I}$ satisfy Assumption 1, that cost functions $\{C_\Omega^i\}_{i \in I}$ satisfy Assumptions 2, 3, and 4, and that the associated divergences $\{D^i\}_{i \in I}$ are RZ-monotone as in Definition 5. Then an s -measurable TSBNE of the game under endogenous information (Definition 3) exists.*

Proof. See the appendix, 7.6. □

The proof is straightforward and is similar to that for Lemma 3. If agents' information costs are RZ-monotone, then agents optimally choose actions that, conditional on (s, \bar{a}) , are independent of both r and z . As a result, if they face an s -measurable \bar{a} function, they will best-respond with a policy whose mean action is indeed s -measurable. We now consider a specific example.

4.3 Example: Beauty Contest with Kullback-Leibler Divergence.

We use an example to show how one may construct an s -measurable equilibria. Consider the simple beauty contest game with Kullback-Leibler (KL) Divergence. The agent's cost function is given by

$$C_\Omega^i(\nu^j, \nu_0) = \int_\omega \pi^j(\omega) D_{KL}(\nu_\omega^j || \nu_0) d\omega = \int_\omega \pi^j(\omega) \left[\sum_{s \in S, r \in R, z \in Z} \int_{\bar{A}} \nu_\omega^j(s, r, z, \bar{a}) \ln \left(\frac{\nu_\omega^j(s, r, z, \bar{a})}{\nu_0(s, r, z, \bar{a})} \right) d\bar{a} \right] d\omega$$

where $\pi^j(\omega) = \sum_{s \in S, r \in R} \int_{\bar{A}} \nu^j(\omega | s, r, z, \bar{a}) \nu_0(s, r, z, \bar{a}) d\bar{a}$.

Consider the agent's optimality conditions for her signal structure presented in equation (12). The derivative of the KL cost function with respect to the signal structure is given by:

$$\frac{\partial C_\Omega^i(\nu^j, \nu_0)}{\partial \nu^j(\omega | s, r, z, \bar{a})} = [\ln(\nu^j(\omega | s, r, z, \bar{a})) - \ln(\pi^j(\omega))] \nu_0(s, r, z, \bar{a})$$

As a result, with KL divergence the optimality conditions in (12) reduce to:

$$u(\alpha(\omega), \bar{a}, s) - \ln \left(\frac{\nu^j(\omega | s, r, z, \bar{a})}{\pi^j(\omega)} \right) = \kappa(s, r, z, \bar{a})$$

for all (s, r, z, \bar{a}) such that $\nu_0(s, r, z, \bar{a}) > 0$. Again consider any two signals $\omega, \omega' \in \Omega$. Rearranging this equation for $\omega, \omega' \in \Omega$ conditional on the same (s, r, z, \bar{a}) yields:

$$u(\alpha(\omega), \bar{a}, s) - u(\alpha(\omega'), \bar{a}, s) = \ln \left(\frac{\nu^j(\omega | s, r, z, \bar{a})}{\pi^j(\omega)} \right) - \ln \left(\frac{\nu^j(\omega' | s, r, z, \bar{a})}{\pi^j(\omega')} \right). \quad (19)$$

The left hand side of this equation—the difference in payoffs—depends on neither r nor z by the assumption that these shocks are not payoff-relevant. Furthermore, the right hand side—the difference in costs—is identical for all $r \in R, z \in Z$. Therefore, conditional on (s, \bar{a}) , the agent’s optimal choice of probability mass on $\omega, \omega' \in \Omega$ must be independent of (r, z) .

The agent finds it optimal to choose a signal structure that pays absolutely no attention to “noise” in the form of (r, z) . Note that this result in fact has nothing to do with the assumed beauty-contest game payoffs; it is simply a consequence of KL divergence and the assumption that r and z are payoff-irrelevant. KL divergence is RZ-invariant which implies it is also RZ-monotone. From Lemma 3 this implies that the minimally-informative signal structure is always the least costly. Combining this with the fact that both (r, z) are irrelevant for payoffs gives us the result.

Next we consider what this signal choice implies for equilibrium. Recall that $\alpha(\omega)$ denotes the beauty contest best response given in (11). By the law of large numbers,

$$\int_{\omega} \alpha(\omega) \mu_{RZ}^*(\omega|s, \bar{a}) d\omega = \hat{\alpha}(s, \bar{a}).$$

That is, aggregation over actions given the agents’ optimal signal structure induces aggregate action $\hat{\alpha}(s, \bar{a})$. An equilibrium aggregate action function $\bar{\alpha} \in \bar{\mathcal{A}}$ is a function defined by the fixed point

$$\bar{\alpha}(s, r, z) = \{\bar{a} \in \mathbb{R} | \bar{a} = \hat{\alpha}(s, \bar{a})\}$$

and must therefore be s-measurable.

4.4 RZ non-monotonicity

The sufficient conditions in Proposition 3 are stronger than necessary. We do not need divergences to be RZ-monotone on *all* priors. Instead, it is sufficient for divergences to be RZ-monotone on all priors that may occur in equilibrium given an s-measurable $\bar{\alpha}$ function.

We define the space of all probability measures that may be generated on $S \times R \times Z \times \bar{\mathcal{A}}$ by some pair $(\mu_0, \bar{\alpha})$ where $\bar{\alpha} \in \bar{\mathcal{A}}$ is s-measurable as

$$\mathcal{V}_0^s = \{\nu \in \mathcal{V}_0 : \exists \mu_0 \in \mathcal{U}_0 \text{ and s-measurable } \bar{\alpha} \in \bar{\mathcal{A}} \text{ s.t. } \nu = \phi_{\bar{\mathcal{A}}} \{\mu_0, \bar{\alpha}\}\}.$$

Therefore, we say the set $\mathcal{V}_0^s \subseteq \mathcal{V}_0$ is the set of all priors that may be **generated by s-**

measurable equilibria. In light of these definitions, we may weaken the conditions in Proposition 3 as follows.

Proposition 4. *Assume that payoffs $\{u^i\}_{i \in I}$ satisfy Assumption 1, that cost functions $\{C_\Omega^i\}_{i \in I}$ satisfy Assumptions 2, 3, and 4, and that the associated divergences $\{D^i\}_{i \in I}$ are RZ-monotone on all priors $\nu_0 \in \mathcal{V}_0^s \subseteq \mathcal{V}_0$. Then an s-measurable TSBNE of the game under endogenous information (Definition 3) exists.*

Proof. Follows from Proposition 3. □

That is, we do not require RZ-monotonicity on all priors, only those that could be generated from an s-measurable $\bar{\alpha}$ function. Weakening the conditions in Proposition 3 is instructive as it allows us to begin considering the converse.

We next define the “opposite” of RZ-monotonicity, relying on the “only-if” aspect of Lemma (3). Consider any informative signal structure $\mu_{RZ} \in \mathcal{U}_{RZ, \Omega}$; we say that a signal structure $\mu_{RZ} \in \mathcal{U}_{RZ, \Omega}$ is informative if the distribution of signal realizations depends on the values of $(s, \bar{\alpha})$. We will say that a cost function C_Ω is “generically RZ-non-monotone” if the minimally informative signal structure $\bar{\nu} \{\mu_{RZ}\}$ is *not* the least-costly of all the signal structures that coarsen to μ_{RZ} , except at isolated points. Our use of the term generic follows Geanakoplos and Polemarchakis [1986] and Farhi and Werning [2016].

Definition 6. A cost function C_Ω satisfying Assumptions 2, 4, and 5 is **generically RZ-non-monotone** if for all $\nu_0 \in \mathcal{V}_0$ and all signal structures $\mu_{RZ} \in \mathcal{U}_{RZ, \Omega}$ such that μ_{RZ} is informative, there exists a $\nu' \in \mathcal{V}'_\Omega \{\mu_{RZ}\}$ such that

$$C_\Omega^i(\nu', \nu_0) < C_\Omega^i(\bar{\nu} \{\mu_{RZ}\}, \nu_0)$$

except on a possibly empty set of isolated priors $\nu_0 \in \mathcal{V}_0$.

We have defined generic RZ-non-monotonicity as a property of the cost functions C (as opposed to of the divergences D) purely for convenience. Note that RZ-monotonicity and generic RZ-non-monotonicity are not exhaustive classes of cost functions; cost functions might exhibit R-monotonicity for some priors but not others. We have little to say about whether s-measurable equilibria will or will not exist in this case.

Armed with this definition, let us first state our result and then give some examples to explain its intuition.

Proposition 5. Fix the set of types I and their action spaces $\{A^i\}_{i \in I}$. Let $\{C^i\}_{i \in I}$ be a set of information costs satisfying Assumptions 2, 3, and 5. Assume that for some type $i^* \in I$, C^{i^*} is generically RZ-non-monotone on priors $\nu_0 \in \mathcal{V}_0^s \subseteq \mathcal{V}_0$. Then there exist payoff functions $\{u^i\}_{i \in I}$ satisfying Assumption 1 such that s-measurable TSBNE exist only on a possibly empty subset of isolated points in the space of priors, \mathcal{V}_0 .

Proof. See the appendix, 7.7. □

This result is stated in terms of the existence of a utility function. The reason for this caveat is the possibility that non-invariance affects the distribution of agents' actions, but not the mean. This possibility opens the door to having equilibria in which individual agents condition their actions on r , but the aggregate action function is nevertheless s-measurable. We construct an example of a utility function that eliminates this possibility.

Using our definition of generic RZ-non-monotonicity, we construct an example satisfying our assumptions for which s-measurable equilibria do not exist. In our example, the result is intuitive: if agents choose to receive a noisy public signal r , their mistakes will be correlated, and this will introduce non-payoff-relevant volatility into equilibrium aggregate actions.

4.5 Example: Generically RZ-non-monotone costs

We construct an example of a generically RZ-non-monotone cost function and use this to explain why we interpret r as a common signal.

Take any signal structure (conditional probability) $\nu^j \in \mathcal{V}_\Omega$ and prior $\nu_0 \in \mathcal{V}_0$; together these induce a joint distribution given by $\hat{\nu}^j(\omega, s, r, z, \bar{a}) = \nu^j(\omega|s, r, z, \bar{a}) \nu_0(s, r, z, \bar{a})$. Let $\nu_{r\omega} \{\nu^j, \nu_0\}$ denote the probability measure on $S \times Z \times \bar{A}$ conditional on $r \in R, \omega \in \Omega$ induced by $\{\nu^j, \nu_0\}$

$$\nu_{r\omega}^j(s, z, \bar{a}) = \nu_{r\omega} \{\nu^j, \nu_0\}(s, z, \bar{a}) = \frac{\hat{\nu}^j(\omega, s, r, z, \bar{a})}{\pi^j(r, \omega)} = \frac{\nu^j(\omega|s, r, z, \bar{a}) \nu_0(s, r, z, \bar{a})}{\pi^j(r, \omega)}$$

where $\pi^j(r, \omega)$ denotes the unconditional probability of state (r, ω) induced by $\{\nu^j, \nu_0\}$. That is $\pi^j(r, \omega) = \sum_{s \in S, z \in Z} \int_{\bar{A}} \nu^j(\omega|s, r, z, \bar{a}) \nu_0(s, r, z, \bar{a}) d\bar{a}$. Similarly, let

$$\nu_{0,r}(s, z, \bar{a}) = \frac{\nu_0(s, r, z, \bar{a})}{\pi(r)}$$

denote the probability measure on $S \times Z \times \bar{A}$ conditional on $r \in R$ induced by the prior.

We use these objects to build a divergence from a pair of KL divergences. Consider the posterior-separable cost function associated with the divergence

$$D(\nu_\omega^j || \nu_0) = \theta_1 D_{KL}(\nu_\omega^j || \nu_0) + \theta_2 \sum_{r \in R} \pi(r) D_{KL}(\nu_{r\omega}^j || \nu_{0,r})$$

where θ_1 and θ_2 are positive constants. When $\theta_2 = 0$, this divergence is simply the KL divergence. When $\theta_2 > 0$, there is an extra penalty for having a distribution conditional on r under ν_ω^j that deviates from the distribution conditional on r under the prior ν_0 . In the limit as $\theta_2 \rightarrow \infty$, the cost to learn about r remains unchanged, but it becomes impossible to learn anything aside from r . In this case, the signal structure $\bar{\nu} \{ \mu_{RZ} \}$ would be infinitely costly for all informative signal structures μ_{RZ} . Consequently, it is preferable for the agent to receive signals about r even though it is *not* payoff-relevant. Even away from this limit, it will generally be cheaper for the agent to choose signal structures that vary in r instead of using the signal structure $\bar{\nu} \{ \mu_{RZ} \}$.

We interpret this cost function as capturing the idea of a noisy public signal. It is comparatively cheap for agents to observe r as opposed to receiving their own signals about the fundamentals s , although they will not perfectly observe either variable. As a result, if such an agent were confronted with an aggregate action function $\bar{\alpha}$ that was s -measurable, she would nevertheless choose to receive signals about r . Because the realization of r is common across all agents, this will introduce correlated errors into the agents' inference about s and hence their actions, resulting in best responses that are not s -measurable.

The two key features of this example, r being a cheap way of learning about s and r being common across agents, are in our view the defining features of what is usually meant by a noisy public signal. Note that this interpretation does not rely on the particular functional form of our example; any cost function which makes it cheaper to learn about s by learning about r than to learn about s directly will have this property.

The results and their limitations highlight different possible interpretations of $r \in R$. We have suggested interpreting $r \in R$ as a common signal. When this common signal is about the fundamentals in a way that is directly relevant for the agent's actions (the generic R -non-monotonicity case), we should expect that the aggregate action is influenced by the common signal. In contrast, if the common signal is about how informative other signals are, we might expect it to affect the variance of individual actions but perhaps not to affect the mean (the possibility ruled out in Proposition 5). Yet another possibility is that the common

signal is a common signal about $z \in Z$. In this case, we might expect R-non-monotonicity for most priors, but R-monotonicity in the special case in which the prior is s-measurable. What all of these examples illustrate is the close connection between the cost functions C^i and the nature of the common signal $r \in R$.

Consider our four example cost functions (mutual information, Tsallis, neighborhoods, and LLR). As we have already shown, with mutual information, because it is invariant in the information-geometric sense, is clearly RZ-monotone. Consequently, with mutual information, s-measurable equilibria will exist. In contrast, Tsallis entropy is generically R-non-monotone, and will generate s-measurable equilibria only if the conditional distribution of r given s is uniform on its support for all $s \in S$ (see [Caplin et al. \[2018\]](#)). The neighborhood cost function is generically R-non-monotone except in the case that the topology imposes no structure on R . Similarly, the LLR cost function is generically R-non-monotone unless the distance function β is not in fact a function of r . For all of these cost functions, we generally expect that non-fundamental volatility will occur in equilibrium. Our interpretation is that using the standard cost function (mutual information) often implicitly rules out the possibility of common signals, which may or may not be justified depending on the economic setting.

We next turn to questions of efficiency, and the connection between the cost functions and informational externalities.

5 Efficiency

We begin by defining the planner's problem for our game with exogenous information. The planner takes can choose the strategies of the agents along with their signal structure to maximize welfare, but is constrained in that she may not simply endow agents with information or move information from one agent to another. We define the efficient allocation as follows.

Definition 7. A strategy profile $\{\nu_A^i \in \mathcal{V}_A^i\}_{i \in I}$ and aggregate action profile $\bar{a} \in \bar{\mathcal{A}}$ of the game with exogenous information (Definition 2) is constrained efficient if, for some strictly positive Pareto-weights $\{\lambda^i\}_{i \in I}$, the strategies and are the solution to the problem

$$\sup_{\{\nu_A^i \in \mathcal{V}_{BD}^i(\nu_\Omega^i)\}_{i \in I}, \bar{a} \in \bar{\mathcal{A}}} \sum_{i \in I} \lambda^i \sum_{s \in S, r \in R, z \in Z} \int_{\bar{\mathcal{A}}} \left[\int_{A^i} u^i(a^j, \bar{a}, s) \nu_A^j(a^j | s, r, z, \bar{a}) da^j \right] \nu_0\{\mu_0, \bar{a}\}(s, r, z, \bar{a}) d\bar{a}$$

subject to constraint of mean consistency,

$$\int_{A^i} a^j \nu_A^i(a^j | s, r, z, \bar{\alpha}^i(s, r, z)) da^j = \bar{\alpha}^i(s, r, z) \quad \forall i \in I, s \in S, r \in R, z \in Z, \text{ s.t. } \mu_0(s, r, z) > 0.$$

Note that these conditional distributions are indexed by types i , which enforces type-symmetry as a constraint on the planner.

It is immediately apparent from this definition that a TSBNE with exogenous information (Definition 2) will not necessarily be efficient. Agents do not internalize the impact that their actions have on aggregate actions, which in turn affect the welfare of others. Of course, pecuniary externalities of this form do not necessarily lead to inefficiency—as the classic welfare theorems demonstrate.

To characterize these externalities, we focus on utility functions that are strictly concave and differentiable, and are such that all optimal actions are interior.¹⁸ The following proposition describes these externalities.

Proposition 6. *Assume that Assumption 1 holds, and in addition that for all $i \in I$ and $s \in S$, $u^i(a, \bar{a}, s)$ is strictly concave in a for all $\bar{a} \in \bar{A}$. Let $(\{\nu_A^{i*}\}_{i \in I}, \bar{\alpha}^*)$ denote a solution to the planner’s problem with strictly positive Pareto-weights $\{\lambda^i\}_{i \in I}$, and assume in addition that the support of ν_A^{i*} is a subset of the interior of A^i for all $i \in I$. Then $(\{\nu_A^{i*}\}_{i \in I}, \bar{\alpha}^*)$ are a TSBNE of the game with exogenous information if and only if, for all $i \in I$, (s, r, z) such that $\mu_0(s, r, z) > 0$, and $a^j \in A^i$ such that $\pi(a^j; \nu_A^{i*}, \nu_0\{\mu_0, \bar{\alpha}^*\}) > 0$,*

$$\sum_{s \in S, r \in R, z \in Z} \int_{\bar{A}} \left(\sum_{i' \in I} \int_{A^{i'}} \lambda^{i'} \frac{\partial u^{i'}(a^l, \bar{a}, s)}{\partial \bar{a}^i} \nu_A^{i'*}(a^l | s, r, z, \bar{a}) da^l \right) \frac{\nu_A^{i*}(a^j | s, r, z, \bar{a}) \nu_0\{\mu_0, \bar{\alpha}^*\}(s, r, z, \bar{a})}{\pi(a^j; \nu_A^{i*}, \nu_0\{\mu_0, \bar{\alpha}^*\})} d\bar{a} = 0.$$

Proof. See the appendix, 7.8. □

We think of these externalities as “distributed information externalities,” an interpretation we elaborate on in our examples. The basic idea is that agents in the game impact each other through the effects that their actions have on the aggregate action, which enters other agents’ utility functions. We use the moniker “distributed information” to indicate an externality may arise even in settings, like a market economy, in which under full information there are no externalities. The key idea here is that the impact of the price change is evaluated not under the prior ν_0 but under the posterior an agent of type i after taking

¹⁸The externalities exist without these assumptions, but are harder to characterize.

action a^j ,

$$\frac{\nu_A^{i*}(a^j|s, r, z, \bar{a}) \nu_0\{\mu_0, \bar{\alpha}^*\}(s, r, z, \bar{a})}{\pi(a^j; \nu_A^{i*}, \nu_0\{\mu_0, \bar{\alpha}^*\})}.$$

This posterior contains information that is not generally available to other agents (even of the same type). As a result, agent j might realize that agent j' is making a mistake, even if agent j' is behaving optimally given the realization of his own signals. The planner would like agent j to take the mistake of agent j' into account, and change her own behavior to increase the utility of agent j' . In the single-type, linear-quadratic-Gaussian environment studied by [Angeletos and Pavan \[2007\]](#), how the planner would like agent j to respond to a signal is determined by a comparison between the social and private degrees of strategic complementarity between the actions of agents j and j' . Our [Proposition 6](#) is a generalization of this result to our setting, along with a re-characterization of the externality as being related to distributed information.

The primary focus of our analysis is not on these externalities, but on externalities related to the cost of acquiring information. To isolate these externalities, we will assume away the “distributed information pecuniary externalities” just described. Our next assumption formalizes this.

Assumption 6. *For all $\bar{\alpha} \in \bar{\mathcal{A}}$ and $\{\nu_A^i \in \mathcal{V}_{\bar{\mathcal{A}}, \mathcal{A}}^i\}_{i \in I}$ satisfying the mean-consistency condition (equation (15)), there exists a set of strictly positive Pareto-weights $\{\lambda^i\}_{i \in I}$ such that, for all $(s, r, z) \in S \times R \times Z$ such that $\mu_0(s, r, z) > 0$,*

$$\bar{\alpha}(s, r, z) \in \arg \max_{\bar{a} \in \bar{\mathcal{A}}} \sum_{k \in I} \int_{A^k} \lambda^i u^k(a^l, \bar{a}, s) \nu_A^k(a^l|s, r, z, \bar{a}) da^l. \quad (20)$$

This assumption is a restriction on the utility functions and price functions. In the single-agent case (when I is a singleton), we can explicitly characterize the functional form of the utility function.

Proposition 7. *Suppose that there is a single type of agent, $I = \{0\}$, that [Assumptions 1](#) and [6](#) hold, and in addition that u^0 is continuously twice-differentiable in its first argument. Then there exists a convex function $H : S \times A^0 \rightarrow \mathbb{R}$ and function $G : S \times A^0 \rightarrow \mathbb{R}$ such that*

$$u^0(a, \bar{a}, s) = G(a; s) + H(\bar{a}; s) + (a - \bar{a}) \cdot \nabla H(\bar{a}; s) \quad (21)$$

where $\nabla H(\bar{a}; s)$ denotes the gradient of H with respect to its first argument.

Proof. See the appendix, 7.9. □

The proof of this result relies on an analogy: the utility function u^0 is, in some sense, like a divergence between a and \bar{a} . That is, under Assumption 6, utility is highest when an individual agent's actions match the aggregate action (this follows from considering degenerate ν_A^k distributions). Armed with this analogy and the results of Banerjee et al. [2005], we derive the result. Note that this result applies only to the single-type case, which has been the focus of most of the beauty contest literature. The multi-type case is more complex because of the possibility of terms in utility functions that “cancel out” across types.

Our next proposition demonstrates that Assumption 6 is a sufficient condition for efficiency under exogenous information.

Proposition 8. *Assume that Assumption 1 holds. If Assumption 6 holds, then a constrained efficient TSBNE of the game with exogenous information (Definition 7) exists for all possible exogenous signal structures $\{\nu_\Omega^i \in \mathcal{V}_{\bar{A}, \Omega}\}_{i \in I}$.*

Proof. See the appendix, 7.10. □

The proof follows from the observation that, if we relax the planner's problem of Definition 7 by ignoring the mean-consistency constraint, the planner will (without loss of generality) nevertheless choose to satisfy mean-consistency by Assumption 6. Consequently, the conditions characterizing the planner's choices for ν_A^i are identical to those of private agents.

Assumption 6 is not necessary for efficiency, because it applies to all conditional action distributions $\nu_A^i \in \mathcal{V}_{\bar{A}, \Omega}$, even ones that (for example) place positive probability on dominated actions. The stronger conditions of Proposition 6 allow us to provide an only-if result for the efficiency of equilibria under exogenous information.

We now turn to the question of constrained efficiency with endogenous information. We begin by defining the planner's problem.

Definition 8. A TSBNE of the game with endogenous information (Definition 3) is constrained efficient if, for some strictly positive Pareto-weights $\{\lambda^i\}_{i \in I}$, the strategies $\{\nu_A^i \in \mathcal{V}_A^i\}_{i \in I}$ and aggregate action profile $\bar{a} \in \bar{\mathcal{A}}$ are the solution to the problem

$$\begin{aligned} \sup_{\{\nu_A^i \in \mathcal{V}_A^i\}_{i \in I}, \bar{a} \in \bar{\mathcal{A}}} & \sum_{i \in I} \lambda^i \sum_{s \in S, r \in R, z \in Z} \int_{\bar{\mathcal{A}}} \left[\int_{A^i} u^i(a^j, \bar{a}, s) \nu_A^j(a^j | s, r, z, \bar{a}) da^j \right] \nu_0\{\mu_0, \bar{a}\}(s, r, z, \bar{a}) d\bar{a} \\ & - \sum_{i \in I} \lambda^i C_A^i(\nu_A^i, \nu_0\{\mu_0, \bar{a}\}) \end{aligned}$$

subject to constraint of mean consistency,

$$\int_{A^i} a^j \nu_A^i(a^j | s, r, z, \bar{\alpha}(s, r, z)) da^j = \bar{\alpha}^i(s, r, z) \quad \forall i \in I, s \in S, r \in R, z \in Z, s.t. \mu_0(s, r, z) > 0.$$

Mirroring our definitions of equilibrium, the key difference between the endogenous information case and exogenous information case is that with exogenous information, ν_A^i must lie in the convex set $\mathcal{V}_{BD}^i(\nu_\Omega^i)$, but there is no utility cost, whereas in the endogenous information there are no restrictions on ν_A^i but there is a utility cost C^i .

This analysis also makes clear why, at least potentially, a new type of externality has been introduced into the economy. The convex sets $\mathcal{V}_{BD}^i(\nu_\Omega^i)$ did not depend on the aggregate action function $\bar{\alpha}$; this is the typical assumption of one makes with exogenously given signals. In contrast, the C^i function in general does depend on the $\bar{\alpha}$ function, and therefore creates another channel by which the actions of one agent affect the welfare of others. Note that this distinction is not really about flexibility in information choice but instead about whether information can be acquired about the endogenous actions of others. One could choose to study economies with fixed, exogenously given signal structures who signal distributions were sensitive to the actions of other agents. Classic models of moral hazard with observable signals (e.g. [Holmstrom et al. \[1979\]](#)) are a leading example of this kind of economy.

However, there are cost functions for which these externalities vanish. In keeping with the theme of the previous sections, we define “invariance with respect to $\bar{\mathcal{A}}$ ” along the same lines as our previous definition of invariance with respect to Φ_Z . We begin again by defining a coarsening operation, $\gamma_{\bar{\mathcal{A}}} : \mathcal{V}_0 \rightarrow \mathcal{U}_0$ by, for all $s \in S, z \in Z, r \in R$,

$$\gamma_{\bar{\mathcal{A}}} \{\nu\}(s, r, z) = \int_{\bar{\mathcal{A}}} \nu(s, r, z, \bar{a}) d\bar{a}.$$

This coarsening operation throws away information about the conditional distribution of \bar{a} given (s, r, z) .

Furthermore, recall that the aggregate action function $\bar{\alpha}$ defines a particular embedding from \mathcal{U}_0 to \mathcal{V}_0 through the function $\phi_{\bar{\mathcal{A}}} \{\mu_0, \bar{\alpha}\}$ defined in (2), and that the set $\bar{\mathcal{A}}$ defines the set of all embeddings. Armed with a coarsening operation and a set of embeddings, we define invariance with respect to $\bar{\mathcal{A}}$.

Definition 9. A divergence $D : \mathcal{V}_0 \times \mathcal{V}_0 \rightarrow \mathbb{R}_+$ is invariant with respect to $\bar{\mathcal{A}}$ if for all

$\bar{\alpha}, \bar{\alpha}' \in \bar{\mathcal{A}}$ and $\nu_0, \nu_1 \in \mathcal{V}_0$,

$$D(\nu_1 || \nu_0) \geq D(\phi_{\bar{\mathcal{A}}} \{\gamma_{\bar{\mathcal{A}}} \{\nu_1\}, \bar{\alpha}\} || \phi_{\bar{\mathcal{A}}} \{\gamma_{\bar{\mathcal{A}}} \{\nu_0\}, \bar{\alpha}\}) = D(\phi_{\bar{\mathcal{A}}} \{\gamma_{\bar{\mathcal{A}}} \{\nu_1\}, \bar{\alpha}'\} || \phi_{\bar{\mathcal{A}}} \{\gamma_{\bar{\mathcal{A}}} \{\nu_0\}, \bar{\alpha}'\}), \quad (22)$$

If all agents have cost functions associated with divergences that are invariant with respect to $\bar{\mathcal{A}}$ on (r,s)-measurable priors, then there are no externalities related to the cost of information. To see this, consider the posteriors over \mathcal{V}_0 and \mathcal{U}_0 conditional on ω using Bayes' rule, $\nu_\omega \{\nu^j, \nu_0\}$ and $\mu_\omega \{\nu^j, \nu_0\}$, and note that

$$\gamma_{\bar{\mathcal{A}}}(\nu_\omega(\nu^j, \nu_0\{\mu_0, \bar{\alpha}\})) = \mu_\omega(\nu^j, \nu_0\{\mu_0, \bar{\alpha}\})$$

and that

$$\nu_\omega(\nu^j, \nu_0\{\mu_0, \bar{\alpha}\}) = \phi_{\bar{\mathcal{A}}}(\mu_\omega(\nu^j, \nu_0\{\mu_0, \bar{\alpha}\}), \bar{\alpha}).$$

Now suppose that ν^j does not depend on \bar{a} ($\nu^j \in \mathcal{V}_{\bar{\mathcal{A}}, \Omega}$). Under this assumption, which is without loss of generality by Lemma 2, the unconditional probabilities $\pi^j\{\nu^j, \nu_0\}$ and posteriors $\mu_\omega(\nu^j, \nu_0\{\mu_0, \bar{\alpha}\})$ do not depend on \bar{a} . If in addition the cost function C^i is associated with a divergence that is invariant with respect to $\bar{\mathcal{A}}$, changing how the posteriors μ_ω are embedded into \mathcal{V}_0 does not affect the divergence. Consequently, $C_\Omega^i(\nu^j, \nu_0\{\mu_0, \bar{\alpha}\}) = C_\Omega^i(\nu^j, \nu_0\{\mu_0, \bar{\alpha}'\})$ for all $\nu^j \in \mathcal{V}_{\bar{\mathcal{A}}, \Omega}$ and $\bar{\alpha}, \bar{\alpha}' \in \bar{\mathcal{A}}$.

In other words, there is no channel by which one agents' actions affect another agents' cost of information. Consequently, if the economy is constrained efficient with exogenous information (which is guaranteed by Assumption 6), it will be constrained efficient with endogenous information.

At least in theory, invariance with respect to $\bar{\mathcal{A}}$ is not necessary for efficiency, for three reasons. First, it could be the case that multiple types in I have cost functions that are not invariant with respect to $\bar{\mathcal{A}}$, and yet a Pareto-weighted sum of the cost functions is invariant with respect to $\bar{\mathcal{A}}$. Second, it could be the case that the value of \bar{a} satisfying mean-consistency is always the minimizer of the cost function, even though the cost function in general depends on \bar{a} . Third, it could be the case that there are externalities associated with information acquisition and externalities under exogenous information, but these externalities happen to cancel. The following assumption summarizes these possibilities in a single condition.

Assumption 7. For all $\mu_0 \in \mathcal{U}_0$ and (r,s)-measurable $\bar{\alpha} \in \bar{\mathcal{A}}$, $\{\nu_A^i \in \mathcal{V}_{\bar{\mathcal{A}}, \Omega}\}_{i \in I}$ satisfying the mean-consistency condition (equation (15)), there exists a set of strictly positive Pareto-

weights $\{\lambda^i\}_{i \in I}$ such that

$$\bar{\alpha} \in \arg \max_{\bar{\alpha}' \in \bar{\mathcal{A}}} \sum_{i \in I} \lambda^i \sum_{s \in S, r \in R, z \in Z} \left[\int_{A^i} u^i(a^j, \bar{\alpha}'(s, r, z), s) \nu_A^i(a^j | s, r, z, \bar{\alpha}'(s, r, z)) da^j \right] \mu_0(s, r, z) - \sum_{i \in I} \lambda^i C_A^i(\nu_A^i, \nu_0\{\mu_0, \bar{\alpha}\}).$$

This assumption essentially states that no externalities exist. It will hold by construction if Assumption 6 holds and all of the cost functions C^i are invariant with respect to $\bar{\mathcal{A}}$, as well as in the three cases discussed above. The following proposition argues that this assumption (combined with assumptions that guarantee existence of an (r,s)-measurable equilibrium by Proposition 2)

Proposition 9. *Assume that Assumptions 1 and 7 hold, and that, for all $i \in I$, the cost functions C^i satisfy Assumptions 2, 3, 4, and 5. Then there exists a TSBNE of the game with endogenous information (Definition 3) that is also the solution to the planner's problem (Definition 8).*

Proof. See the appendix, 7.11. □

To study the inefficient equilibria, we introduce a notion of generic non-invariance. We use the notation $\nabla_{\bar{\alpha}} C(\nu^j, \nu_0\{\mu_0, \bar{\alpha}\})$ to denote the gradient of C with respect to $\bar{\alpha}$, if it exists.

Definition 10. A cost function C^i for type $i \in I$ is **generically non-invariant on $\bar{\mathcal{A}}$** if, for all informative $\nu^i \in \mathcal{V}_{\bar{\mathcal{A}}, \Omega}$ and all $\mu_0 \in \mathcal{U}_0, \bar{\alpha} \in \bar{\mathcal{A}}$ such that $\bar{\alpha}^i$ and μ^i satisfy mean-consistency, except on a possibly empty set of isolated values of $(\nu^j, \mu_0, \bar{\alpha})$,

$$\nabla_{\bar{\alpha}} C_A^i(\nu^i; \nu_0\{\mu_0, \bar{\alpha}\}) \neq \vec{0}.$$

With generic non-invariance, there will in general be an interaction between agent's actions and other agents' cost of information. This leads to inefficiency, even when the economy is efficient under exogenous information. We prove this in the special case of a single type.

Proposition 10. *Suppose that there is a single type of agent, $I = \{0\}$, and fix the action space A^0 . Let C^0 be an information cost satisfying Assumptions 2, 3, and 5, and assume in addition that C^0 is generically non-invariant on $\bar{\mathcal{A}}$. Then there exists a price function p and*

utility function u^0 satisfying Assumptions 1 and 6 such that a constrained-efficient TSBNE under exogenous information exists only on a possibly empty subset of isolated points in the space of priors, \mathcal{U}_0 .

Proof. See the appendix, 7.12. □

Like Proposition 7, the difficulty of extending this result to multiple types arises because of the possibility that individual types may have generically non-invariant cost functions and yet the effects of a change in the aggregate action can “cancel out” across types.

Turning to our four examples, it is immediately apparent the the KL divergence, again because of its general invariance, is invariant with respect to $\bar{\mathcal{A}}$ on (r,s)-measurable priors. Consequently, if the economy is efficient with exogenous information, it will also be efficient under endogenous information acquisition if all agents have a KL divergence cost function. Perhaps more surprisingly, the same property holds true for the Tsallis cost function, despite that cost function not being invariant in the information-geometric sense. The reason this property holds is (r,s)-measurability: because the distribution of \bar{a} conditional on (r, s) is degenerate, the function $H_{TS,\rho}(\nu)$ in fact depends only on the distribution of (r, s) , and hence is unaffected by changes in \bar{a} . In contrast, the neighborhood based cost function, assuming the topology imposes some structure on $\bar{\mathcal{A}}$, will be for most topologies be generically non-invariant. Similarly, the LLR function, assuming the distance function depends on \bar{a} , will be generically non-invariant for most distance functions. This difference between the Tsallis and neighborhood/LLR cost functions arises because, although none of these cost functions are invariant in the information-geometric sense, only the latter two incorporate the idea that the values of \bar{a} have an inherent meaning.

6 Conclusion

In this paper, we have explored the relationship between cost functions and the properties of equilibria in large games with rationally inattentive agents. Under the assumption of posterior separability, we have demonstrated the close connection between certain kinds of invariance and whether or not the equilibrium is efficient/exhibits non-fundamental volatility. We have interpreted these forms of invariance as describing whether or not it is possible to learn directly about the actions of others and whether or not public signals are available to the agent. Efficiency holds only when it is not possible to learn directly about the actions of others and there are no externalities under exogenous information; efficiency under exogenous

information is related to what we call “dispersed information pecuniary externalities.” The class of utility functions that rules out such externalities has a particular functional form that generalizes the characterization of ? to games that are not necessarily linear-quadratic-Gaussian.

References

- Manuel Amador and Pierre-Olivier Weill. Learning from prices: Public communication and welfare. *Journal of Political Economy*, 118(5):866–907, 2010. [6](#)
- Shun-ichi Amari and Hiroshi Nagaoka. *Methods of information geometry*, volume 191. American Mathematical Soc., 2007. [1](#), [1](#), [2.2](#)
- George-Marios Angeletos and Jennifer La’O. Dispersed information over the business cycle: Optimal fiscal and monetary policy. *MIT mimeo*, 2008. [6](#)
- George-Marios Angeletos and Jennifer La’O. Noisy business cycles. In *NBER Macroeconomics Annual 2009, Volume 24*, pages 319–378. University of Chicago Press, 2010. [1](#)
- George-Marios Angeletos and Jennifer La’O. Sentiments. *Econometrica*, 81(2):739–779, 2013. [1](#)
- George-Marios Angeletos and Jennifer La’O. Optimal monetary policy with informational frictions. *Journal of Political Economy*, 2020. [5](#)
- George-Marios Angeletos and Alessandro Pavan. Efficient use of information and social value of information. *Econometrica*, 75(4):1103–1142, 2007. [1](#), [1](#), [1](#), [5](#)
- George-Marios Angeletos and Alessandro Pavan. Policy with dispersed information. *Journal of the European Economic Association*, 7(1):11–60, 2009. [6](#)
- George-Marios Angeletos and Karthik Sastry. Inattentive economies: General equilibrium and welfare theorems. *MIT mimeo*, 2019. [1](#), [1](#)
- George-Marios Angeletos, Luigi Iovino, and Jennifer La’O. Efficiency and policy with endogenous learning. 2015. [6](#)

- Arindam Banerjee, Xin Guo, and Hui Wang. On the optimality of conditional expectation as a bregman predictor. *IEEE Transactions on Information Theory*, 51(7):2664–2669, 2005. 5, 7.9
- David Blackwell. Equivalent comparisons of experiments. *The annals of mathematical statistics*, 24(2):265–272, 1953. 2, 3.1
- Andrew Caplin and Mark Dean. Revealed preference, rational inattention, and costly information acquisition. *American Economic Review*, 105(7):2183–2203, 2015. 2.2, 11
- Andrew Caplin, Mark Dean, and John Leahy. Rationally inattentive behavior: Characterizing and generalizing Shannon entropy. *Unpublished manuscript*, 2018. 1, 1, 1, 2.2, 2.2, 2.2, 2.4, 2.4, 4.5
- Nikolai Nikolaevich Chentsov. *Statistical decision rules and optimal inference*. Number 53. American Mathematical Soc., 1982. 2.2, 14
- Luca Colombo, Gianluca Femminis, and Alessandro Pavan. Information acquisition and welfare. *The Review of Economic Studies*, 81(4):1438–1483, 2014. 2, 3, 1
- Mark Dean and Nathaniel Neligh. Experimental tests of rational inattention. Technical report, Working Paper, Columbia University, 2018. 1
- Tommaso Denti. Unrestricted information acquisition. *WP, Cornell University*, 2015. 1, 1, 7, 2.2
- Tommaso Denti. Network effects in information acquisition. *Princeton University*, 2017. 7
- Emmanuel Farhi and Iván Werning. A theory of macroprudential policies in the presence of nominal rigidities. *Econometrica*, 84(5):1645–1704, 2016. 4.4
- Drew Fudenberg and Jean Tirole. Game theory, 1991. *Cambridge, Massachusetts*, 393(12):80, 1991. 8
- John D Geanakoplos and Heraklis M Polemarchakis. Existence, regularity, and constrained suboptimality of competitive allocations. *Essays in Honor of Kenneth J. Arrow: Volume 3, Uncertainty, Information, and Communication*, 3:65, 1986. 4.4
- Sanford J Grossman and Joseph E Stiglitz. On the impossibility of informationally efficient markets. *The American economic review*, 70(3):393–408, 1980. 1

- Benjamin Hébert. Moral hazard and the optimality of debt. *The Review of Economic Studies*, 85(4):2214–2252, 2018. [1](#), [2.2](#)
- Benjamin Hébert and Michael Woodford. Rational inattention in continuous time. *Unpublished manuscript*, September 2018a. [1](#), [2.2](#), [11](#), [2.2](#)
- Benjamin Hébert and Michael Woodford. Information costs and sequential information sampling. Technical report, National Bureau of Economic Research, 2018b. [1](#), [2.4](#), [2.4](#), [16](#)
- Christian Hellwig and Laura Veldkamp. Knowing what others know: Coordination motives in information acquisition. *The Review of Economic Studies*, 76(1):223–251, 2009. [1](#)
- Bengt Holmstrom et al. Moral hazard and observability. *Bell journal of Economics*, 10(1):74–91, 1979. [1](#), [5](#)
- Stephen Morris and Hyun Song Shin. Social value of public information. *American Economic Review*, 92(5):1521–1534, 2002. [1](#)
- David P Myatt and Chris Wallace. Endogenous information acquisition in coordination games. *The Review of Economic Studies*, 79(1):340–374, 2011. [1](#)
- Luciano Pomatto, Philipp Strack, and Omer Tamuz. The cost of information. *arXiv preprint arXiv:1812.04211*, 2018. [1](#), [2.4](#), [2.4](#)
- Alexandros Rigos et al. Flexible information acquisition in large coordination games. Technical report, 2018. [7](#)
- Christopher A. Sims. Implications of rational inattention. *Journal of Monetary Economics*, 50(3):665 – 690, 2003. ISSN 0304-3932. doi: [https://doi.org/10.1016/S0304-3932\(03\)00029-1](https://doi.org/10.1016/S0304-3932(03)00029-1). URL <http://www.sciencedirect.com/science/article/pii/S0304393203000291>. Swiss National Bank/Study Center Gerzensee Conference on Monetary Policy under Incomplete Information. [1](#), [1](#), [2.4](#)
- Michael Woodford. Imperfect common knowledge and the effects of monetary policy. *Knowledge, Information, and Expectations in Modern Macroeconomics: In Honor of Edmund S. Phelps*, 2003. [1](#)

Appendix

7 Proofs

7.1 Proof of Lemma 1

For any signal structure ν' ,

$$C^i(\nu'; \nu_0) = \int_{\Omega^i} \pi'(\omega) D^i(\nu'_\omega || \nu_0) d\omega$$

where

$$\pi'(\omega) = \pi\{\nu', \nu_0\}(\omega) = \sum_{s \in S, r \in R, z \in Z} \int_{\bar{A}} \nu'(\omega|s, r, z, \bar{a}) \nu_0(s, r, z, \bar{a}) d\bar{a},$$

$$\nu'_\omega(s, r, z, \bar{a}) = \nu_\omega\{\nu', \nu_0\}(s, r, z, \bar{a}) = \frac{\nu'(\omega|s, r, z, \bar{a}) \nu_0(s, r, z, \bar{a})}{\pi\{\nu', \nu_0\}(\omega)},$$

and D^i is invariant in Φ_Z in the sense described by Assumption 5.

By definition of $\nu' \in \bar{\mathcal{V}}_\Omega\{\mu_Z\}$,

$$\mu_Z(\omega|s, r, \bar{a}) = \frac{\sum_{z \in Z} [\nu'(\omega|s, r, z, \bar{a}) \nu_0(s, r, z, \bar{a})]}{\gamma_Z(\nu_0(s, r, z, \bar{a}))}$$

Given this signal structure, the unconditional probability of observing $\omega \in \Omega$ satisfies

$$\begin{aligned} \pi\{\nu', \nu_0\}(\omega) &= \sum_{s \in S, r \in R, z \in Z} \int_{\bar{A}} \nu'(\omega|s, r, z, \bar{a}) \nu_0(s, r, z, \bar{a}) d\bar{a} \\ &= \sum_{s \in S, r \in R} \int_{\bar{A}} \left[\sum_{z \in Z} \nu'(\omega|s, r, z, \bar{a}) \nu_0(s, r, z, \bar{a}) \right] d\bar{a} \\ &= \sum_{s \in S, r \in R} \int_{\bar{A}} [\mu_Z(\omega|s, r, \bar{a}) \gamma_Z(\nu_0(s, r, z, \bar{a}))] d\bar{a} \end{aligned}$$

and therefore

$$\pi\{\nu', \nu_0\} = \pi\{\bar{\nu}, \nu_0\}, \quad \forall \nu' \in \bar{\mathcal{V}}_\Omega\{\mu_Z\}. \quad (23)$$

Further note that for all $\omega \in \Omega$,

$$\begin{aligned}\gamma_Z(\nu'_\omega(s, r, z, \bar{a})) &= \frac{\sum_{z \in Z} [\nu'_\omega(\omega|s, r, z, \bar{a}) \nu_0(s, r, z, \bar{a})]}{\pi\{\nu', \nu_0\}(\omega)}, \\ &= \frac{\mu_Z(\omega|s, r, \bar{a}) \gamma_Z(\nu_0(s, r, z, \bar{a}))}{\pi\{\bar{\nu}, \nu_0\}(\omega)}\end{aligned}$$

and hence is identical for all $\nu' \in \bar{\mathcal{V}}_\Omega\{\mu_Z\}$.

Define an embedding ϕ_Z such that

$$\phi_Z(\gamma_Z(\nu_0)) = \nu_0;$$

this embedding's corresponding conditional distribution function is given by

$$\hat{\phi}_Z(z|s, r, \bar{a}) = \frac{\nu_0(s, r, z, \bar{a})}{\gamma_Z(\nu_0(s, r, z, \bar{a}))}.$$

Applying this embedding to $\gamma_Z(\nu'_\omega)$,

$$\begin{aligned}\phi_Z(\gamma_Z(\nu'_\omega(s, r, z, \bar{a}))) &= \hat{\phi}_Z(z|s, r, \bar{a}) \gamma_Z(\nu'_\omega(s, r, z, \bar{a})) \\ &= \frac{\nu_0(s, r, z, \bar{a})}{\gamma_Z(\nu_0(s, r, z, \bar{a}))} \frac{\mu_Z(\omega|s, r, \bar{a}) \gamma_Z(\nu_0(s, r, z, \bar{a}))}{\pi\{\bar{\nu}, \nu_0\}(\omega)} \\ &= \frac{\mu_Z(\omega|s, r, \bar{a}) \nu_0(s, r, z, \bar{a})}{\pi\{\bar{\nu}, \nu_0\}(\omega)}\end{aligned}$$

Therefore

$$\phi_Z(\gamma_Z(\nu'_\omega)) = \bar{\nu}_\omega \quad \forall \nu' \in \bar{\mathcal{V}}_\Omega\{\mu_Z\}.$$

By Assumption 5,

$$D(\nu'_\omega || \nu_0) \geq D(\phi_Z(\gamma_Z(\nu'_\omega)) || \phi_Z(\gamma_Z(\nu_0))) = D(\bar{\nu}_\omega || \nu_0).$$

Combining this with (23), we therefore have that

$$C^i(\nu'; \nu_0) \geq C^i(\bar{\nu}, \nu_0) \quad \forall \nu' \in \bar{\mathcal{V}}_\Omega\{\mu_Z\}.$$

as required.

7.2 Proof of Lemma 2

We begin with the following observation: suppose that for some $\nu_A^j, \nu_A^{j'} \in \mathcal{V}_A^i$ and $\bar{\alpha} \in \bar{\mathcal{A}}$, and all $(a^j, s, r, z) \in A^i \times S \times R \times Z$,

$$\nu_A^j(a^j|s, r, z, \bar{\alpha}(s, r, z)) = \nu_A^{j'}(a^j|s, r, z, \bar{\alpha}(s, r, z)).$$

Then, by Assumption 4,

$$C_A^i(\nu_A^{j'}, \nu_0\{\mu_0, \bar{\alpha}\}) = C_A^i(\nu_A^j, \nu_0\{\mu_0, \bar{\alpha}\}).$$

This result follows from the fact that signal distributions conditional on zero-probability events have no impact on the unconditional signal probabilities or posteriors and posterior-separability.

Consequently, it is without loss of generality to suppose that the signals ν_A^j do not depend on the value of $\bar{\alpha}$ conditional on (s, r, z) . In other words, we can restrict attention to $\nu_A^j \in \mathcal{V}_{\bar{A}, A}^i \subset \mathcal{V}_A^i$ in both the endogenous and exogenous cases.

7.3 Proof of Proposition 1

Invoking the results of Lemma 2, it is without loss of generality to assume that $\nu_A^i \in \mathcal{V}_{\bar{A}, A}^i$ for all $i \in I$.

To handle both the exogenous information case and the endogenous information case, define $\bar{\mathcal{V}}_A^i = \mathcal{V}_{\bar{A}, A}^i$ in the endogenous information case, and $\bar{\mathcal{V}}_A^i = \mathcal{V}_{BD}^i(\nu_\Omega^i)$ in the exogenous information case. We begin by noting that $\mathcal{V}_{\bar{A}, A}^i$ is a finite (by the finiteness of $S \times R \times Z$) set of measures on the compact subsets of R^L (i.e. A^i), by Assumption 1 (which assumes compactness for A^i). Consequently, by Prokhorov's theorem, using the topology of weak convergence, $\mathcal{V}_{\bar{A}, A}^i$ is compact. By equation (13), $\mathcal{V}_{BD}^i(\nu_\Omega^i)$ is convex and compact (in the topology of weak convergence) subset of $\mathcal{V}_{\bar{A}, A}^i$. Therefore, maxima exist, and note also that the set of feasible policies is non-empty and does not depend on $\bar{\alpha}$.

To adapt the proof to the exogenous information case, suppose that in this case there is

an information cost function $C_A^i(\nu_A^j, \nu_0\{\mu_0, \bar{\alpha}\}) = 0$. Individual optimality requires that

$$\nu_A^i \in \max_{\nu_A^j \in \bar{\mathcal{V}}_A^i} \sum_{s \in S, r \in R, z \in Z} \int_{\bar{A}} \left[\int_{A^i} u^i(a^j, \bar{a}, s) \nu_A^j(a^j | s, r, z, \bar{a}) da^j \right] \nu_0\{\mu_0, \bar{\alpha}\}(s, r, z, \bar{a}) d\bar{a} - C_A^i(\nu_A^j, \nu_0\{\mu_0, \bar{\alpha}\}),$$

and mean consistency requires that

$$\int_{A^i} a^j \nu_A^i(a^j | s, r, z, \bar{\alpha}^i(s, r, z)) da^j = \bar{\alpha}^i(s, r, z) \quad \forall i \in I, s \in S, r \in R, z \in Z, s.t. \mu_0(s, r, z) > 0.$$

We apply the theorem of the maximum and Kakutani's fixed point theorem in the usual fashion. Observe by the continuity of u^i (Assumption 1), and by Assumption 3 (the continuity of C^i), that the objective function of the individual agent's problem is continuous in $(\nu_A^i, \bar{\alpha})$. Consequently, we can invoke the theorem of the maximum.

The optimal policy correspondence $\mathcal{A}^{i*} : \bar{\mathcal{A}} \rightarrow \bar{\mathcal{V}}_A^i$ is non-empty, upper semi-continuous, and compact-valued. By the concavity of the objective function (due the convexity of the cost function, Assumption 2), the optimal policy correspondence is convex.

Define $\bar{\mathcal{A}}^i$ as the set of possible aggregate actions $\bar{\alpha}^i : S \times R \times Z \rightarrow A^i$ for type $i \in I$. From the optimal policy correspondence, define the function $f^i : \bar{\mathcal{V}}_A^i \rightarrow \bar{\mathcal{A}}^i$ by

$$f^i(\nu_A^i) = \int_{A^i} a^j \nu_A^i(a^j | s, r, z, \bar{\alpha}(s, r, z)) da^j.$$

Observe that f^i is continuous and linear in ν_A^i , and does not in fact depend on $\bar{\alpha}$.

Define the correspondence $F : \bar{\mathcal{A}} \rightarrow \bar{\mathcal{A}}$ by composing the correspondences \mathcal{A}^{i*} and the functions f^i , and taking the product space of the resulting sets:

$$F(\bar{A}) = \prod_{i \in I} f^i \circ \mathcal{A}^{i*}(\bar{A}).$$

By the upper semi-continuity of \mathcal{A}^{i*} and continuity of f^i , F is upper semi-continuous. By the non-emptiness of \mathcal{A}^{i*} , F is non-empty. By the convexity of \mathcal{A}^{i*} and the linearity of f^i , F is convex.

By the finiteness of $S \times R \times Z$ and the fact that $\bar{A} \subseteq \mathbb{R}^{L \times |I|}$, $\bar{\mathcal{A}}$ is isomorphic to a subset of $\mathbb{R}^{|S| \times |R| \times |Z| \times L \times |I|}$. Consequently, by Kakutani's fixed point theorem, there exists a fixed

point of the correspondence F . This fixed point, by construction, constitutes a TSBNE.

7.4 Proof of Proposition 2

The proof is essentially identical to our existence proof (Proposition 1), and we will refer to the proof of that proposition rather than repeat most of the arguments. Let $\bar{\mathcal{A}}_{RS} \subset \bar{\mathcal{A}}$ denote the subset of α functions that are (r,s)-measurable. Let $\mathcal{V}_{AZ,A}^i \subset \mathcal{V}_A^i$ denote the set of signal structures whose distributions do not in fact depend on either z or \bar{a} .

By Lemma 1 (characterizing invariance on Φ_Z), the optimality policy correspondences \mathcal{A}^{i*} (defined in the proof of Proposition 1) are mappings from $\bar{\mathcal{A}}_{RS}$ to $\mathcal{V}_{AZ,A}^i$. By construction, the functions f^i defined in the proof of Proposition 1 map conditional distributions in $\mathcal{V}_{AZ,A}^i$ to functions that are measurable on (r, s) . Consequently, the mapping F defined in the proof of Proposition 1 is a map from $\bar{\mathcal{A}}_{RS}$ to $\bar{\mathcal{A}}_{RS}$.

The arguments for the upper semi-continuity, non-emptiness, and convexity of F apply unchanged. It follows that a fixed point in $\bar{\mathcal{A}}_{RS}$ exists, and this fixed point constitutes an (r,s)-measurable equilibrium.

7.5 Proof of Lemma 3

Part (i). The “if” part of the proof repeats the proof of Lemma 1. We simply coarsen in both r and z to the (s, \bar{a}) space and then embed back to the larger (s, r, z, \bar{a}) space using the η_{RZ} function.

Part (ii). We prove the “only if” by contradiction. Suppose there exists $\nu_0, \nu_1 \in \mathcal{V}_0$ with $\gamma_{RZ} \{\nu_1\} \ll \gamma_{RZ} \{\nu_0\}$ such that

$$D(\nu_1 || \nu_0) < D(\eta_{RZ}(\nu_1, \nu_0) || \nu_0).$$

Define $\Omega = \{\omega_1, \omega_2\}$ and define, for $\epsilon > 0$ sufficiently small, the signal structure $\mu_{RZ,\epsilon}$ as follows

$$\mu_{RZ,\epsilon}(\omega_1 | s, \bar{a}) = \delta(\omega_1) \epsilon \frac{\gamma_{RZ} \{\nu_1\}(s, r, z, \bar{a})}{\gamma_{RZ} \{\nu_0\}(s, r, z, \bar{a})} = \delta(\omega_1) \epsilon \frac{\sum_{r \in R, z \in Z} \nu_1(s, r, z, \bar{a})}{\sum_{r \in R, z \in Z} \nu_0(s, r, z, \bar{a})}$$

and

$$\mu_{RZ,\epsilon}(\omega_2|s, \bar{a}) = \delta(\omega_2) \left(1 - \epsilon \frac{\gamma_{RZ} \{\nu_1\}(s, r, z, \bar{a})}{\gamma_{RZ} \{\nu_0\}(s, r, z, \bar{a})} \right) = \delta(\omega_2) \left(1 - \epsilon \frac{\sum_{r \in R, z \in Z} \nu_1(s, r, z, \bar{a})}{\sum_{r \in R, z \in Z} \nu_0(s, r, z, \bar{a})} \right),$$

where $\delta(\cdot)$ is the Dirac delta function. Note that

$$\eta_{RZ} \{\nu_1, \nu_0\}(s, r, z, \bar{a}) = \nu_0(s, r, z, \bar{a}) \frac{\sum_{r \in R, z \in Z} \nu_1(s, r, z, \bar{a})}{\sum_{r \in R, z \in Z} \nu_0(s, r, z, \bar{a})}$$

Therefore

$$\mu_{RZ,\epsilon}(\omega_1|s, \bar{a}) = \delta(\omega_1) \epsilon \frac{\eta_{RZ} \{\nu_1, \nu_0\}(s, r, z, \bar{a})}{\nu_0(s, r, z, \bar{a})}$$

and

$$\mu_{RZ,\epsilon}(\omega_2|s, \bar{a}) = \delta(\omega_2) \left(1 - \epsilon \frac{\eta_{RZ} \{\nu_1, \nu_0\}(s, r, z, \bar{a})}{\nu_0(s, r, z, \bar{a})} \right),$$

Next, consider the minimally-informative signal structure $\bar{\nu} \{\mu_{RZ,\epsilon}\}$ is given by

$$\bar{\nu} \{\mu_{RZ,\epsilon}\}(\omega|s, r, z, \bar{a}) = \mu_{RZ,\epsilon}(\omega|s, \bar{a}).$$

for all $s \in S, r \in R, z \in Z, \bar{a} \in \bar{A}, \omega \in \Omega$. Posteriors of the minimally-informative signal structure are given by

$$\begin{aligned} \nu_{\omega_1} \{\bar{\nu} \{\mu_{RZ,\epsilon}\}, \nu_0\} &= \eta_{RZ} \{\nu_1, \nu_0\} \\ \nu_{\omega_2} \{\bar{\nu} \{\mu_{RZ,\epsilon}\}, \nu_0\} &= \frac{1}{1-\epsilon} \nu_0 - \frac{\epsilon}{1-\epsilon} \eta_{RZ} \{\nu_1, \nu_0\}. \end{aligned}$$

Now consider instead an alternative signal structure

$$\begin{aligned} \nu'_\epsilon(\omega_1|s, r, z, \bar{a}) &= \delta(\omega_1) \epsilon \frac{\nu_1(s, r, z, \bar{a})}{\nu_0(s, r, z, \bar{a})}, \\ \nu'_\epsilon(\omega_2|s, r, z, \bar{a}) &= \delta(\omega_2) \left(1 - \epsilon \frac{\nu_1(s, r, z, \bar{a})}{\nu_0(s, r, z, \bar{a})} \right). \end{aligned}$$

By construction $\nu'_\epsilon \in \mathcal{V}'_\Omega \{\mu_{RZ,\epsilon}\}$. Posteriors given this signal structure are given by

$$\begin{aligned} \nu_{\omega_1} \{\nu'_\epsilon, \nu_0\} &= \nu_1 \\ \nu_{\omega_2} \{\nu'_\epsilon, \nu_0\} &= \frac{1}{1-\epsilon} \nu_0 - \frac{\epsilon}{1-\epsilon} \nu_1. \end{aligned}$$

The cost of signal structure $\bar{\nu} \{\mu_{RZ,\epsilon}\}$ is given by,

$$C(\bar{\nu} \{\mu_{RZ,\epsilon}\}, \nu_0) = \epsilon D(\eta_{RZ} \{\nu_1, \nu_0\} || \nu_0) + (1 - \epsilon) D\left(\nu_0 - \frac{\epsilon}{1 - \epsilon} (\eta_{RZ} \{\nu_1, \nu_0\} - \nu_0) \Big\| \nu_0\right),$$

and the cost of signal structure ν'_ϵ is likewise given by,

$$C(\nu'_\epsilon, \nu_0) = \epsilon D(\nu_1 || \nu_0) + (1 - \epsilon) D\left(\nu_0 - \frac{\epsilon}{1 - \epsilon} (\nu_1 - \nu_0) \Big\| \nu_0\right).$$

Now consider the difference between these two cost structures:

$$f(\epsilon) \equiv C(\nu'_\epsilon, \nu_0) - C(\bar{\nu} \{\mu_{RZ,\epsilon}\}, \nu_0).$$

By assumption, $D(\nu' || \nu_0)$ is differentiable with respect to ν' at $\nu' = \nu_0$, and the gradient must be zero by the definition of the divergence. Therefore, $f(\epsilon)$ is differentiable with respect to ϵ at $\epsilon = 0^+$ and

$$f'(\epsilon) |_{\epsilon=0^+} = D(\nu_1 || \nu_0) - D(\eta_{RZ} \{\nu_1, \nu_0\} || \nu_0) < 0.$$

But we must have that

$$C(\nu'_\epsilon, \nu_0) - C(\bar{\nu} \{\mu_{RZ,\epsilon}\}, \nu_0) \geq 0$$

for all feasible $\epsilon \geq 0$, and $f(0) = 0$, contradicting the premise that $D(\nu_1 || \nu_0) < D(\eta_{RZ}(\nu_1, \nu_0) || \nu_0)$.

7.6 Proof of Proposition 3

The proof is essentially identical to our existence proof (Proposition 1) and the proof of Proposition 2, and we will refer to the existence proof (Proposition 1) rather than repeat most of the arguments. Let $\bar{\mathcal{A}}_S \subset \bar{\mathcal{A}}$ denote the subset of α functions that are s-measurable. Let $\mathcal{V}_{\bar{A}RZ,A}^i \subset \mathcal{V}_A^i$ denote the set of signal structures who distributions do not in fact depend on any of r, z, \bar{a} .

By Lemma 3 (characterizing RZ-monotonicity), the optimality policy correspondences \mathcal{A}^{i*} (defined in the proof of Proposition 1) are mappings from $\bar{\mathcal{A}}_S$ (inducing s-measurable priors) to $\mathcal{V}_{\bar{A}RZ,A}^i$. By construction, the functions f^i defined in the proof of Proposition 1 map conditional distributions in $\mathcal{V}_{\bar{A}RZ,A}^i$ to functions that are measurable on s . Consequently, the mapping F defined in the proof of Proposition 1 is a map from $\bar{\mathcal{A}}_S$ to $\bar{\mathcal{A}}_S$.

The arguments for the upper semi-continuity, non-emptiness, and convexity of F apply unchanged from the proof of Proposition 1. It follows that a fixed point in $\bar{\mathcal{A}}_S$ exists, and this fixed point constitutes an s-measurable equilibrium.

7.7 Proof of Proposition 5

The utility functions for all types except i^* are irrelevant for the argument, and can be chosen arbitrarily.

Suppose that the type i^* has a utility function that is strictly convex on A^{i^*} for all (p, s) . By the convexity and compactness of A^{i^*} , extreme points exist, and by the strict convexity of the utility function, all $a \in A^{i^*}$ that are not extreme points are strictly dominated by a mixed strategy over extreme points. Consequently, the agents of type i^* will only choose actions that are extreme points. Note that this example essentially converts the set of actions for type i^* into a finite set.

Now suppose that in a neighborhood of some $\mu_0 \in \mathcal{U}_0$, there are at least two priors $\mu_1, \mu_2 \in \mathcal{U}_0$ such that an s-measurable equilibrium exists, and let α_1 and α_2 denote the corresponding s-measurable functions. By the definition of generic RZ-non-monotonicity, the agents of type i^* must respond to (μ_1, α_1) and (μ_2, α_2) by choosing some $\mu_1^i, \mu_2^i \in \mathcal{U}_A^i$ such that, for at least one of the μ_1^i, μ_2^i , the distribution of $a^j \in A^{i^*}$ depends on r . Suppose without loss of generality this holds for μ_1^i .

Because this distribution has support only on the extreme points of A^{i^*} , the mean value

$$\int_{A^{i^*}} a^j \mu_1^{i^*}(a^j | s, r, z) da^j$$

is unique for all conditional distributions $\mu_1^{i^*}(a^j | s, r, z)$. Consequently, because $\mu_1^{i^*}(a^j | s, r, z) \neq \mu_1^{i^*}(a^j | s, r', z)$ for some $r, r' \in R$, the mean actions $\bar{a}^{i^*}(s, r, z)$ and $\bar{a}^{i^*}(s, r', z)$ must differ, contradicting the existence of an s-measurable equilibrium. Therefore, if an s-measurable equilibrium exists, it must be an isolated value of μ_0 .

7.8 Proof of 6

Observe first that, by $\nu_A^i \in \mathcal{V}_{BD}^i(\nu_\Omega^i)$, the conditional distributions of ν_A^i do not depend on \bar{a} . Consequently, the Lagrangean version of the planner's problem can be written as

$$\begin{aligned} & \sup_{\{\nu_A^i \in \mathcal{V}_{BD}^i(\nu_\Omega^i)\}_{i \in I}, \bar{\alpha} \in \bar{A}} \inf_{\psi \in \mathbb{R}^{|S| \times |R| \times |Z| \times |I| \times L}} \sum_{i \in I} \lambda^i \sum_{s \in S, r \in R, z \in Z} \left[\int_{A^i} u^i(a^j, \bar{\alpha}(s, r, z), s) \nu_A^i(a^j | s, r, z, \bar{\alpha}(s, r, z)) da^j \right] \mu_0 \\ & + \sum_{i \in I} \sum_{l=1}^L \sum_{s \in S, r \in R, z \in Z} \lambda^i \mu_0(s, r, z) \psi_l^i(s, r, z) [\bar{\alpha}_l^i(s, r, z) - \int_{A^i} a_l^j \nu_A^i(a^j | s, r, z, \bar{\alpha}(s, r, z)) da^j]. \end{aligned}$$

Note that we have scaled the multiplier by $\mu_0(s, r, z)$ to denote that the policy need not hold for (s, r, z) not in the support of μ_0 .

By strict concavity, there is a single action a^j for each signal realizations ω^j . Recall that the proposition assumes interior solutions.

The first-order condition for $a_l^i(\omega^j)$ is

$$\begin{aligned} & \lambda^i \sum_{s \in S, r \in R, z \in Z} \int_{\bar{A}} \frac{\partial u^i(a, \bar{a}, s)}{\partial a_l} \Big|_{a=a^j} \nu_A^{i*}(a^j | s, r, z, \bar{\alpha}(s, r, z)) \mu_0(s, r, z) \\ & - \sum_{s \in S, r \in R, z \in Z} \lambda^i \mu_0(s, r, z) \psi_l^i(s, r, z) \nu_A^{i*}(a^j | s, r, z, \bar{\alpha}(s, r, z)) = 0. \end{aligned}$$

The first-order condition for $\bar{\alpha}_l^i(s, r, z)$ is

$$\begin{aligned} & \left(\sum_{i' \in I} \int_{A^{i'}} \lambda^{i'} \frac{\partial u^{i'}(a^l, \bar{a}, s)}{\partial \bar{\alpha}_l^{i'}} \Big|_{\bar{a}=\bar{\alpha}^*(s, r, z)} \nu_A^{i'*}(a^l | s, r, z, \bar{\alpha}(s, r, z)) da^l \right) \mu_0(s, r, z) \\ & + \lambda^i \mu_0(s, r, z) \psi_l^i(s, r, z) = 0. \end{aligned}$$

In contrast, the private FOC for $a_l^i(\omega^j)$ is

$$\sum_{s \in S, r \in R, z \in Z} \frac{\partial u^i(a, \bar{a}, s)}{\partial a_l} \nu_A^i(a^j | s, r, z, \bar{\alpha}(s, r, z)) \mu_0(s, r, z) = 0.$$

By strict concavity, it follows from the private and social FOCs for $a_l^i(\omega^j)$ that efficiency will hold if and only if, for all i and $a^j \in A^i$ such that $\pi(a^j; \nu_A^i, \nu_0) > 0$,

$$\sum_{s \in S, r \in R, z \in Z} \lambda^i \mu_0(s, r, z) \psi_l^i(s, r, z) \nu_A^{i*}(a^j | s, r, z, \bar{\alpha}(s, r, z)) = 0.$$

By the FOC for $\bar{\alpha}_i^i(s, r, z)$, we must have

$$\sum_{s \in S, r \in R, z \in Z} \left(\sum_{i' \in I} \int_{A^{i'}} \lambda^{i'} \frac{\partial u^{i'}(a^l, \bar{a}, s)}{\partial \bar{a}_i^i} \Big|_{\bar{a} = \bar{\alpha}^*(s, r, z)} \nu_A^{i'*}(a^l | s, r, z, \bar{\alpha}(s, r, z)) da^l \right) \mu_0(s, r, z) \nu_A^{i'*}(a^j | s, r, z, \bar{\alpha}(s, r, z))$$

which is the result.

7.9 Proof of Proposition 7

Define the function

$$f(a, \bar{a}; s) = u^0(a, a, s) - u^0(a, \bar{a}, s).$$

Consider the degenerate distribution, for some $(s, r, z) \in S \times R \times Z$ such that $\mu_0(s, r, z) > 0$ and some $a' \in A^0$,

$$\mu_A^0(a | s, r, z) = \delta(a - a'),$$

where $\delta(\cdot)$ is the Dirac delta function. To satisfy mean consistency, we must have $\bar{a}(s, r, z) = a'$. By Assumption 6,

$$a' \in \arg \max_{\bar{a}} u^0(a', p(s, \bar{a}), s)$$

and therefore

$$a' \in \arg \min_{\bar{a}} f(a', \bar{a}; s)$$

and $f(a', a'; s) = 0$ by construction. Consequently, $f(a', a'; s)$ is a weakly positive, continuously differentiable function, continuously twice-differentiable in its first argument. By Assumption 6, for all measures $\mu_A^0(a | s, r, z)$,

$$\int_{A^0} a \mu_A^0(a | s, r, z) da \in \arg \min_{\bar{a} \in A^0} \int_{A^0} f(a, \bar{a}; s) \mu_A^0(a | s, r, z) da.$$

By theorem 4 of [Banerjee et al. \[2005\]](#) (see also the discussion in that paper on restrictions to subspaces of \mathbb{R}^L), it follows that

$$f(a, \bar{a}; s) = H(a; s) - H(\bar{a}; s) - (a - \bar{a}) \cdot \nabla H(\bar{a}; s).$$

Defining

$$G(a; s) = u^0(a, p(s, a), s) - H(a; s)$$

proves the result.

7.10 Proof of Proposition 8

Recall the definition of the planner's problem:

$$\sup_{\{\nu_A^j \in \mathcal{V}_{BD}^i(\nu_\Omega^i)\}_{i \in I}, \bar{\alpha} \in \bar{\mathcal{A}}} \sum_{i \in I} \lambda^i \sum_{s \in S, r \in R, z \in Z} \left[\int_{A^i} u^i(a^j, \bar{\alpha}(s, r, z), s) \nu_A^i(a^j | s, r, z, \bar{\alpha}(s, r, z)) da^j \right] \mu_0(s, r, z),$$

subject to the mean consistency constraint.

Now consider a relaxed version of the problem, without the mean consistency constraint. Let $(\{\nu_A^{i*}\}_{i \in I}, \bar{\alpha}^*)$ denote the solutions to the relaxed problem. By Assumption 6 that there exists some strictly positive Pareto weights such that it is without loss of generality to assume that $\bar{\alpha}^*$ satisfies the mean-consistency condition.

By optimality in the relaxed problem, for each $i \in I$,

$$\nu_A^{i*} \in \arg \max_{\nu_A^j \in \mathcal{V}_{BD}^i(\nu_\Omega^i)} \lambda^i \sum_{s \in S, r \in R, z \in Z} \left[\int_{A^i} u^i(a^j, \bar{\alpha}^*(s, r, z), s) \nu_A^i(a^j | s, r, z, \bar{\alpha}^*(s, r, z)) da^j \right] \mu_0(s, r, z).$$

By the assumption of strictly positive Pareto weights, it immediately follows that $(\{\nu_A^{i*}\}_{i \in I}, \bar{\alpha}^*)$ constitutes a TSBNE.

7.11 Proof of 9

Recall the definition of the planner's problem, applying Lemma 2:

$$\begin{aligned} & \sup_{\{\nu_A^i \in \mathcal{V}_{A,A}^i\}_{i \in I}, \bar{\alpha} \in \bar{\mathcal{A}}} \sum_{i \in I} \lambda^i \sum_{s \in S, r \in R, z \in Z} \left[\int_{A^i} u^i(a^j, \bar{\alpha}(s, r, z), s) \nu_A^i(a^j | s, r, z, \bar{\alpha}(s, r, z)) da^j \right] \mu_0(s, r, z) \\ & - \sum_{i \in I} \lambda^i C_A^i(\nu_A^i, \nu_0\{\mu_0, \bar{\alpha}\}), \end{aligned}$$

subject to the mean consistency constraint.

Now consider a relaxed version of the problem, without the mean consistency constraint. Let $(\{\nu_A^{i*}\}_{i \in I}, \bar{\alpha}^*)$ denote the solutions to the relaxed problem, which exist by the continuity assumptions. By Assumption 7, for the Pareto-weights defined in that assumption, it is without loss of generality to suppose that $\bar{\alpha}^*$ satisfies the mean-consistency condition.

By optimality in the relaxed problem, for each $i \in I$,

$$\begin{aligned} \nu_A^{i*} \in & \sup_{\nu_A^i \in \mathcal{V}_{A,A}^i} \lambda^i \sum_{s \in S, r \in R, z \in Z} \left[\int_{A^i} u^i(a^j, \bar{\alpha}^*(s, r, z), s) \nu_A^i(a^j | s, r, z, \bar{\alpha}^*(s, r, z)) da^j \right] \mu_0(s, r, z) \\ & - \lambda^i C_A^i(\nu_A^i, \nu_0\{\mu_0, \bar{\alpha}^*\}), \end{aligned}$$

By the assumption of strictly positive Pareto weights, it immediately follows that $(\{\nu_A^{i*}\}_{i \in I}, \bar{\alpha}^*)$ constitutes a TSBNE.

7.12 Proof of 10

Suppose that only one dimension of the individual and aggregate actions is relevant for utilities. Suppose that u^0 has the functional form described in Proposition 7, and assume strict concavity with respect to the first argument of the utility function.

The Lagrangean version of the planner's problem is, using 2,

$$\begin{aligned} & \sup_{\nu_A^0 \in \mathcal{V}_{A,A}^0, \bar{\alpha} \in \bar{A}} \inf_{\psi \in \mathbb{R}^{|S| \times |R| \times |Z| \times |I| \times L}} \sum_{s \in S, r \in R, z \in Z} \left[\int_{A^0} u^0(a^j, \bar{\alpha}(s, r, z), s) \mu_A^0(a^j | s, r, z, \bar{\alpha}(s, r, z)) da^j \right] \mu_0(s, r, z) \\ & - C_A^0(\nu_A^0, \nu_0\{\mu_0, \bar{\alpha}\}) \\ & + \sum_{l=1}^L \sum_{s \in S, r \in R, z \in Z} \mu_0(s, r, z) \psi_l^0(s, r, z) [\bar{\alpha}_l^0(s, r, z) - \int_{A^i} a_l^j \mu_A^0(a^j | s, r, z) da^j]. \end{aligned}$$

Note that we have scaled the multiplier by $\mu_0(s, r, z)$ to denote that the policy need not hold for (s, r, z) not in the support of μ_0 .

By strict concavity, there is a single action a^j for each signal realizations ω^j . By adjusting the utility function to penalize actions on the boundaries of A^* , we can guarantee interior solutions, and it is also straightforward to ensure that the support of $\pi(a^j; \nu_A^0, \nu_0)$ is non-degenerate. The first-order condition for $a_l^{i*}(\omega^j)$ is

$$\begin{aligned} & \sum_{s \in S, r \in R, z \in Z} \frac{\partial u^0(a, \bar{\alpha}(s, r, z), s)}{\partial a_l} \Big|_{a=a^j} \nu_A^0(a^j | s, r, z, \bar{\alpha}(s, r, z)) \mu_0(s, r, z) \\ & - \sum_{s \in S, r \in R, z \in Z} \mu_0(s, r, z) \psi_l^0(s, r, z) \nu_A^0(a^j | s, r, z, \bar{\alpha}(s, r, z)) = 0. \end{aligned}$$

The first-order condition for $\bar{\alpha}_l^0(s, r, z)$ is (if the gradient $\nabla_{\bar{\alpha}}$ exists),

$$\begin{aligned} \left(\int_{A^{i^*}} \frac{\partial u^0(a^l, p(s, \bar{\alpha}), s)}{\partial \bar{\alpha}_l^0} \Big|_{\bar{\alpha}=\bar{\alpha}(s, r, z)} \nu_A^0(a^l | s, r, z, \bar{\alpha}(s, r, z)) da^l \right) \mu_0(s, r, z) \\ - \nabla_{\bar{\alpha}_l^0} C_A^0(\nu_A^0, \nu_0\{\mu_0, \bar{\alpha}\}) \{s, r, z\} \\ + \mu_0(s, r, z) \psi_l^0(s, r, z) = 0, \end{aligned}$$

why simplifies by Proposition 21 to

$$\mu_0(s, r, z) \psi_l^0(s, r, z) = \nabla_{\bar{\alpha}_l^0} C_A^0(\nu_A^0, \nu_0\{\mu_0, \bar{\alpha}\}) \{s, r, z\}.$$

In contrast, the private FOC for $a_l^i(\omega^j)$ is

$$\sum_{s \in S, r \in R, z \in Z} \frac{\partial u^0(a, \bar{\alpha}(s, r, z), s)}{\partial a_l} \Big|_{a=a^j} \nu_A^0(a^j | s, r, z, \bar{\alpha}(s, r, z)) \mu_0(s, r, z) = 0.$$

By strict concavity, efficiency requires

$$0 = \sum_{s \in S, r \in R, z \in Z} \nu_A^0(a^j | s, r, z, \bar{\alpha}(s, r, z)) \nabla_{\bar{\alpha}_l^0} C_A^0(\nu_A^0, \nu_0\{\mu_0, \bar{\alpha}\}) \{s, r, z\}$$

for all a^j such that $\pi(a^j; \nu_A^0, \nu_0) > 0$. To extend the argument to the non-differentiable case, observe that efficiency requires the existence of the derivative.

By generic non-invariance on $\bar{\mathcal{A}}$, this is non-zero except at isolated values of $(\nu_A^0, \mu_0, \bar{\alpha})$. The policies that solve the planner's problem are upper semi-continuous in μ_0 by the usual theorem of the maximum arguments, and consequently constrained efficiency holds only at isolated values of μ_0 .