Ambiguity and the Historical Equity Premium

F. Collard
(U. of Adelaide)

S. Mukerji
(Oxford U.)

K. Sheppard
(Oxford U.)

J.M. Tallon
(U. Paris I, PSE)

July 2010

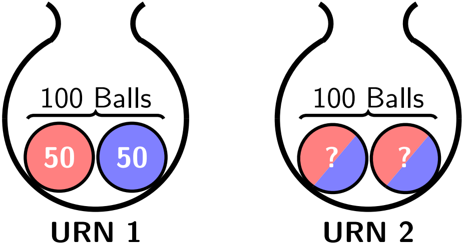
Following

- intuitive arguments of Knight (1921) and Ellsberg (1961),
- pioneering formalizations by Schmeidler (1989) and Gilboa and Schmeidler (1987)

it is now customary in modern decision theory to distinguish between two types of subjectively uncertain belief:

- **Unambiguous belief:** Can be expressed as a probability distribution and is thus like risk.
- **Ambiguous belief:** Cannot be expressed using a single probability distribution $\Longrightarrow$ Ambiguity Aversion.

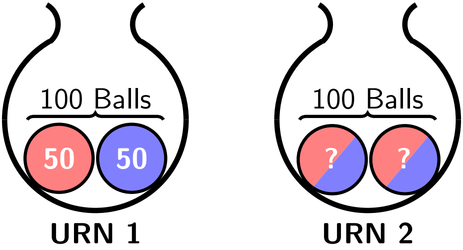
- How *much* of the historically observed equity premium may be accounted for by
  - ambiguity (or, uncertainty) about the probability distribution over future growth in consumption/dividends
  - together with the sensitivity of agents' preferences to this ambiguity?
- Why does this matter? $\Longrightarrow$ Goes back to Ellsberg's paradox



- A DM is endowed with preferences represented by $U(\cdot)$, with $U'(\cdot) > 0$.
- DM is first offered the following bet: Win \$ 100 if **R**, \$ 0 if **B**.
- Experiments suggest Urn 1 $\succ$ Urn 2, SEU theory suggests

$$0.5U(\$100) + 0.5U(\$0) > \pi U(\$100) + (1-\pi)U(\$0)$$

- Therefore: $\pi < 0.5$



- DM is now offered the following bet: Win \$ 100 if **B**, \$ 0 if **R**.
- Experiments suggest Urn 1 $\succ$ Urn 2, SEU theory then suggests

$$0.5U(\$100) + 0.5U(\$0) > (1-\pi)U(\$100) + \pi U(\$0)$$

- Therefore: $\pi > 0.5$

- When DM is endowed with VNM utility, Urn 1 is always chosen, but
  - First bet suggests $\pi < 0.5$
  - Second bet suggests $\pi > 0.5$
- VNM utility is inconsistent with the results of the experiment.
- A single prior cannot express the DM's concern that he knows relatively less about what the "true" prior is in Urn II
- Potential explanation, the DM is ambiguity averse.

- Uncertainty about the (parameters of) probability distribution of stochastic variables of interest.
- Despite all the statistical data available it may not be possible to pin down (some of) the parameters ($\Longrightarrow$ "model or parameter uncertainty")
- In our setting: Ambiguity (or, uncertainty) about the probability distribution over future growth in consumption/dividend **as evident from historical data.**
- beliefs will be statistically grounded: Bayesian updates conditional on relevant, publicly observed data

- Ambiguity (or Knightian uncertainty) aversion is a commonly observed behaviour that is inconsistent with SEU.
- DM are ambiguity averse if
  - they take into account how well they know the relevant odds, and
  - choose actions whose prospects are robust to the imprecision of their knowledge about the odds.

- Pervasive in economic decision making.
- Not particular to the ill informed and less sophisticated DM.
- Often hard to distinguish (on the basis of historical data) between different models providing distinct (stochastic) forecasts of relevant financial variables. (Identification problem)
- Ambiguity averse DMs may well think it is prudent to choose actions that are robust to the uncertainty about the "correct" model/stochastic forecast.
- Similar intuition as the one suggested by Hansen–Sargent's concern for robustness.

Back to the Question

- Quantitative effect of ambiguity/ambiguity aversion (model uncertainty/robustness concerns) on asset price equilibria
- Apply the KMM framework to a Lucas' (1978) tree model
- Does smooth ambiguity provide a good framework to understand the equity premium?
- Yes!

Roadmap

- Preliminaries
- An Ambiguity Aversion version of the Lucas tree model.
- Solution methodology
- Calibration strategy
- Results

- Klibanoff, Marinacci, Mukerji, (Econometrica,2005)
- *Smooth ambiguity preferences* are represented as,

$$V(f) = \int \phi\left(\int u\left(c(s)\right) d\pi_\theta(s)\right) d\mu\left(\theta\right)$$

- Represents the DM's ambiguity/ambiguous belief - it summarizes the DM's subjective uncertainty about the "true" π_θ, probability distribution over contingencies.

- **Idea:** Ambiguity averse DMs prefer acts whose evaluation is more robust to possible variation in probabilities.
- **In KMM:** Translated as an aversion to mean preserving spreads in the induced distribution of expected utilities. (ϕ operator)
- **Technically:**
  - Concavity of the ϕ operator;
  - The DM is more averse to the subjective uncertainty about priors than he is to the risk in lotteries.

- As if the ambiguity averse DM thinks as follows
  - My best guess of the chance that the return distribution is 'p' is 20%.
  - However, this is based on "softer" information than knowing that the chance of a particular outcome in an objective lottery is 20%.
  - Hence, I would like to behave with more caution with respect to the former risk.

- (KMM2, JET 2009) KMM2 extends KMM1 to intertemporal preferences on "consumption plan".
- Recursive Smooth Ambiguity:

$$V_{s^t}(c) = u\left(c\left(s^t\right)\right) + \beta\phi^{-1}\left[\int \phi\left(\int V_{(s^t,n_{t+1})}(C)\, d\pi_\theta\left(n_{t+1}; s^t\right)\right) d\mu\left(\theta \mid s^t\right)\right]$$

- The preferences are dynamically consistent *and* satisfy consequentialism.

An Ambiguity Aversion Lucas' Tree Model

- Standard Lucas tree model up to
  - Ambiguity about the probability distribution over future growth in dividend/consumption growth
  - The representative agent's preferences are sensitive to this ambiguity.
- Roadmap
  - Specification of ambiguity
  - Preferences and returns.

- Two layers of ambiguity
  - Unobservability of a key latent variable (single–ρ model),
  - Uncertainty about the models (persistence and volatility) (two–ρ model).

- Denoted the "single-ρ" model.
- Consumption growth ($g_{c,t+1} = \log(C_{t+1}/C_t)$) and Dividend growth ($g_{d,t+1} = \log(D_{t+1}/D_t)$)

$$g_{c,t+1} = \overline{g}_c + x_{t+1} + \sigma_c \varepsilon_{c,t+1}$$
$$g_{d,t+1} = \overline{g}_d + \psi x_{t+1} + \sigma_d \varepsilon_{d,t+1}$$

  with $\varepsilon_{c,t+1} \perp \varepsilon_{d,t+1}$ and $\varepsilon_{j,t+1} \rightsquigarrow \mathscr{N}(0,1)$, $j = \{c, d\}$.
- $\overline{g}_c$, $\overline{g}_d$, ψ, σ_c and σ_d are known to the agents

- x_t follows an AR(1) process

$$x_{t+1} = \rho x_t + \sigma_x \varepsilon_{x,t+1}$$

with $\varepsilon_{x,t+1} \perp \varepsilon_{c,t+1}$, $\varepsilon_{x,t+1} \perp \varepsilon_{d,t+1}$ and $\varepsilon_{x,t+1} \rightsquigarrow \mathcal{N}(0,1)$.

- ρ, and σ_x are known to the agents
- x_t: Latent variable $\Longleftrightarrow$ **Not** known to the agents
- Since x_t is not observed, the distribution of next period growth rates is uncertain.

$$g_{c,t+1} \rightsquigarrow \mathcal{N}(\overline{g}_c + \rho x_t, \sigma_c^2 + \sigma_x^2)$$
$$g_{d,t+1} \rightsquigarrow \mathcal{N}(\overline{g}_d + \psi\rho x_t, \sigma_c^2 + \psi^2\sigma_x^2)$$

- x_t plays the role of θ in $\pi_\theta(\cdot, s^t)$.

- The agent's beliefs track x_t using observations on $g_{c,t}$ and $g_{d,t}$
- Second order belief $\mu_t : x_t \rightsquigarrow (\widehat{x}_t, P)$
- The update $\widehat{x}_{t+1}$ is given by the Kalman filter ($\Longrightarrow$ Bayesian posterior) Kalman Filter
- Intuition: Agent is uncertain about mean of the distribution of the growth rate.
  - Permanent component, $\widetilde{g}_c$, is known
  - Transitory component, x_t, is never fully learnt, but just tracked $\Longrightarrow$ drives the Knightian uncertainty.

- We further assume that the agents are uncertain about the process of growth rates
- More precisely, parameter uncertain about persistence
  - Low persistence (probability a priori η_t)

$$g_{c,\ell,t+1} = \overline{g}_{c,\ell} + x_{\ell,t+1} + \sigma_{c,\ell}\varepsilon_{c,\ell,t+1}$$
$$g_{d,\ell,t+1} = \overline{g}_{d,\ell} + \psi_\ell x_{\ell,t+1} + \sigma_{d,\ell}\varepsilon_{d,\ell,t+1}$$
$$x_{\ell,t+1} = \rho_\ell x_{\ell,t} + \sigma_{x,\ell}\varepsilon_{x,\ell,t+1}$$

  - High persistence (probability a priori $1 - \eta_t$)

$$g_{c,h,t+1} = \overline{g}_{c,h} + x_{h,t+1} + \sigma_{c,h}\varepsilon_{c,h,t+1}$$
$$g_{d,h,t+1} = \overline{g}_{d,h} + \psi_h x_{h,t+1} + \sigma_{d,h}\varepsilon_{d,h,t+1}$$
$$x_{h,t+1} = \rho_h x_{h,t} + \sigma_{x,h}\varepsilon_{x,h,t+1}$$

- Parameters $\overline{g}_{c,i}$, $\overline{g}_{d,i}$, ψ_i, ρ_i, $\sigma_{c,i}$, $\sigma_{d,i}$ and $\sigma_{x,i}$, $i = \{\ell, h\}$ are known.

- The distribution of growth rates is now a mixture.
- Uncertainty about
  - The true state of the economy
  - How long the current state will persist
- Mixes Kalman filtering ▸ Kalman Filter and learning about η_t ▸ Learning
- Bayesian updating of $\eta_t \Longrightarrow$ time–varying ambiguity.
- Time variations are endogenous.

- Budget constraint

$$P_t^f B_{t+1} + P_t S_{t+1} = B_t + (P_t + D_t) S_t - C_t$$

- Preferences:

$$V(C_t, \widehat{x}_t) = \max\ (1-\beta)u(C_t) + \beta\Phi^{-1}\left(E_{\mu_t}\Phi\left[E_{\pi_{x_t}}\left[V\left(C_{t+1}, \widehat{x}_{t+1}\right)\right]\right]\right)$$

- with

$$C_{t+1} = \exp(g^c_{t+1})C_t \text{ and } D_{t+1} = \exp(g^d_{t+1})D_t$$

- Similar in 2–ρ model

2–ρ Model

- Use the following functional forms

$$U(x) = \frac{x^{1-\gamma} - 1}{1 - \gamma} \text{ and } \Phi(x) = -\frac{\exp(-\alpha x)}{\alpha}$$

- Exponential $\Phi(\cdot)$ simplifies the analysis.
- Non homogeneous value function.

- First order conditions leads to (Bayesian Case)

$$
\begin{aligned}
P_t^f &= \beta E_{\mu_t}\left[\xi_t(x_t) E_{\pi_{x_t}}\left[\frac{U'(C_{t+1})}{U'(C_t)}\right]\right] \\
1 &= \beta E_{\mu_t}\left[\xi_t(x_t) E_{\pi_{x_t}}\left[\left(\frac{P_{t+1}+D_{t+1}}{P_t}\right)\frac{U'(C_{t+1})}{U'(C_t)}\right]\right] \\
1 &= \beta E_{\mu_t}\left[\xi_t(x_t) E_{\pi_{x_t}}\left[R_{t+1}\frac{U'(C_{t+1})}{U'(C_t)}\right]\right]
\end{aligned}
$$

- Standard Euler equations with a *twist*

$$
\xi_t(x_t) = \frac{\Phi'\left(E_{\pi_{x_t}}\left[V(C_{t+1}, \widehat{x}_{t+1})\right]\right)}{E_{\mu_t}\left[\Phi'\left(E_{\pi_{x_t}}\left[V(C_{t+1}, \widehat{x}_{t+1})\right]\right)\right]}
$$

- Recall the twist is given by

$$\xi_t(x_t) = \frac{\Phi'\left(E_{\pi_{x_t}}\left[V(C_{t+1}, \widehat{x}_{t+1})\right]\right)}{E_{\mu_t}\left[\Phi'\left(E_{\pi_{x_t}}\left[V(C_{t+1}, \widehat{x}_{t+1})\right]\right)\right]}$$

- $\xi_t(x_t)$ is a change of measure: Distorts the Bayesian posterior towards hidden states (the π_{x_t}) with lower expected value (as $\Phi'(\cdot) < 0$)
- $\xi_t(x_t)$ depends on $\alpha \iff$ captures the effects of ambiguity aversion.
- $\xi_t(x_t)$ is potentially different at each node. (cannot ascribe an "*as if*" equivalent Bayesian prior for the entire tree)

- Aim: Evaluate quantitatively the model on its ability to account for macro–finance facts
- Strategy:
  - Obtain an approximate solution of the model
  - Constraint ourselves to "*standard*" (admitted) values of risk aversion
  - Plug data in the solution and compute moments for
    - Riskfree rate, $R_t^f = 1/P_t^f$
    - Risky rate (Dividend Claim): $R_t = E_{\mu_t} E_{\pi_{x_t}} \frac{P_{t+1}+D_{t+1}}{P_t}$
    - Risky rate (Consumption Claim): $R_t = E_{\mu_t} E_{\pi_{x_t}} R_{t+1}$
    - Equity Premium

Solution Method

- Solve the Value function problem
- Solve for the price of assets
- Rely on a Galerkin method: assume a solution of the form

$$\Phi_y(X_t) = \exp\left(\sum_{i_c,i_x\in\mathcal{I}} \theta^y_{i_c,i_x} H_{i_c}(\varphi_c(C_t))H_{i_x}(\varphi_x(\widehat{x}_t))\right)$$

  where $X_t \equiv (C_t, \widehat{x}_t)$ denotes the vector of state variables of our single–ρ case and $y \in \{V, R\}$.
- Consumption is unbounded $\Longrightarrow$ Use Hermite polynomials
- Multiple integrals: monomial formula (See Judd [1998], chapter 7.)
- Check accuracy ($\checkmark$)

- ρ is weakly identified
- Dogmatic value for ρ: Bansal–Yaron value ($\rho = 0.85$) (sensitivity analysis with $\rho = 0.90$)
- Dogmatic value for ψ: Bansal–Yaron value ($\psi = 3$)
- In the 2–ρ model: $\rho_h = 0.85$, $\rho_\ell = 0.30$.
- Use consumption growth and dividend growth data + Kalman filter to obtain
    - Estimates of the parameters
    - a time series for $\widehat{x}_t$
- Data span 1930–2007:
    - Process estimated over the 1930–1977 period
    - Use outsample forecasts for the model evaluation (1977–2007)

Parametrization

Forcing Variables

Leverage	ψ	3	3
Constant g_c	$\overline{g}_c$	0.019256	0.019256
Constant g_d	$\overline{g}_d$	0.006241	0.006241
Persistence	ρ	0.85	0.90
Volatility ε_c	σ_c	0.014864	0.013289
Volatility ε_d	σ_d	0.105598	0.106400
Volatility ε_x	σ_x	0.024657	0.025763

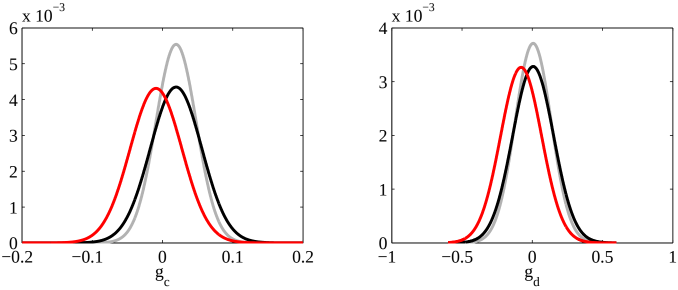
2–ρ Model

- Recall that

$$U(C)=\begin{cases}\frac{C^{1-\gamma}-1}{1-\gamma} & \text{if } \gamma\in\mathbb{R}\backslash\{1\}\\ \log(C) & \text{if } \gamma=1\end{cases} \quad \text{and } \Phi(x)=\exp(-\alpha x),\alpha\geqslant 0$$

- Vary γ. ($\gamma=2.5$ will be used as a Benchmark)
- For each γ, α is such that $E(R_t^f)$ matches the data (2.44%).

Rational Expectations + Bayesian + Twisted



Rational Expectations, Bayesian, Twisted.

$$g_{c,t+1} \rightsquigarrow \mathcal{N}(\overline{g}_c + \rho x_t, \sigma_x^2 + \sigma_c^2) \text{ with } x_t = \widehat{x}_t$$
$$g_{d,t+1} \rightsquigarrow \mathcal{N}(\overline{g}_d + \psi\rho x_t, \psi^2\sigma_x^2 + \sigma_c^2) \text{ with } x_t = \widehat{x}_t$$

	Consumption		Dividend	
	$E(g)$	$\sigma(g)$	$E(g)$	$\sigma(g)$
Rat. Exp.	0.0193	0.0288	0.0063	0.1289
Bayesian	0.0193	0.0367	0.0063	0.1458
Twisted	-0.0093	0.0370	-0.0792	0.1464
	$Sk(g)$	$Ku(g)$	$Sk(g)$	$Ku(g)$
Rat. Exp.	-0.0000	-0.0000	-0.0001	-0.0014
Bayesian	-0.0000	-0.0002	-0.0004	-0.0098
Twisted	0.0007	-0.0001	0.0085	-0.0257

2–ρ Model

Results

Dividend Claims

γ	α	$E(r)$	$E(r-r^f)$	$\sigma(r^f)$	$\sigma(r)$	$\sigma(r-r^f)$
Data		9.83	7.38	2.63	13.94	13.66
Bayes.	r^f =10.03	10.30	0.27	2.40	9.65	9.34
1.00	34.681	10.31	7.87	0.85	6.28	6.20
1.50	22.005	9.99	7.55	1.37	7.28	7.14
2.00	14.198	10.27	7.83	1.93	8.43	8.21
2.50	8.606	10.80	8.36	2.49	9.67	9.37
3.00	4.865	11.47	9.03	3.03	10.98	10.59
$\rho = 0.90$	5.612	10.73	8.29	2.78	10.18	9.77
2–ρ	11.050	11.10	8.66	3.68	13.47	13.68

▸ Consumption Claims

Results

- Bayes–rational information processing + minimal degree of ambiguity aversion $\Longrightarrow$ possible to obtain very significant effects on asset prices.
- Almost enough to explain the data.
- Operates using standard tools of finance: only on "distorted" beliefs
- Distortion based on changes to "*central*" beliefs, not extremes!
- Leads to a natural extension of Bayesian framework that allows for *conditional* pessimism (as the twist shifts the distribution toward the left).

- Builds on Campbell and Shiller (Review of Financial Studies, 1989)
- Riskfree rate:

$$r_t^f \simeq -\log(\beta) + \gamma \widetilde{E}_t\left[g_{c,t+1}\right] - \frac{\gamma^2}{2}\widetilde{\sigma}_c^2$$

where $\widetilde{E}_t[z_{t+1}] = E_{\mu_t}\xi_t(x_t)E_{\pi_{x_t}}[z_{t+1}]$ and $\widetilde{\sigma}_c$ the associated volatility.

- For a given γ, higher α shifts the distribution to the left $\Longrightarrow \widetilde{E}_t[g_{c,t+1}]$ decreases. See Figure
- Higher α leads to a decrease in the riskfree rate.

- Risky rate:

$$r_{t+1} \equiv \log\left(\frac{P_{t+1}+D_{t+1}}{P_t}\right) \simeq \kappa_0 + \kappa_1 p_{t+1} - p_t + g_{d,t+1}$$

  where $p_t = log(P_t/D_t)$.

- Further assume $p_t = A_0 + A_1 x_t$.
- $(\kappa_0, \kappa_1, A_0, A_1)$ must be determined by solving the Euler equation.
- Then, at the solution, we have

$$E_t[r_{t+1}] = \overline{r} - \rho(\psi - \gamma)\widehat{x}_t + E_t[g_{d,t+1}]$$

  where $E_t[z_{t+1}] = E_{\mu_t}\left[E_{\pi_{x_t}}[z_{t+1}]\right]$ is independent of α.

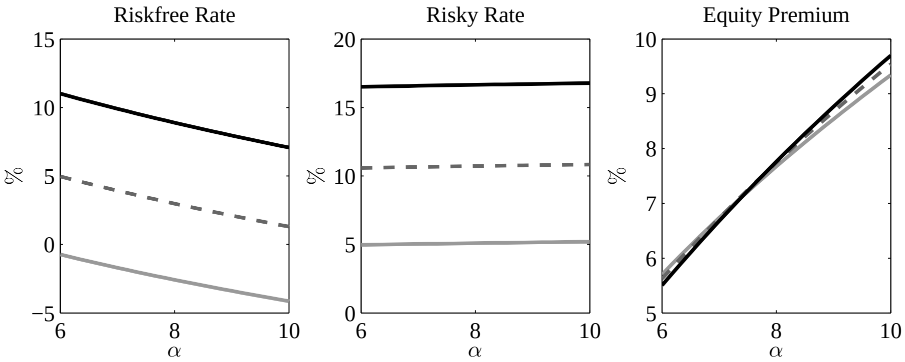
- Equity premium

$$E_t[r_{t+1} - r_t^f] = \bar{r} + \log(\beta) - \rho(\psi - \gamma)\widehat{x}_t + E_t[g_{d,t+1}] - \gamma\widetilde{E}_t[g_{c,t+1}] + \frac{\gamma^2}{2}\widetilde{\sigma}_c^2$$

- For γ given, increasing in α.
- The conditional pessimism the twist triggers explains the equity premium.
- Because the agent dislikes not knowing the risk (the conditional expectation) he faces, he is more cautious and must be rewarded to hold risky assets over riskfree assets.

Results

Comparative statics: Variations in α

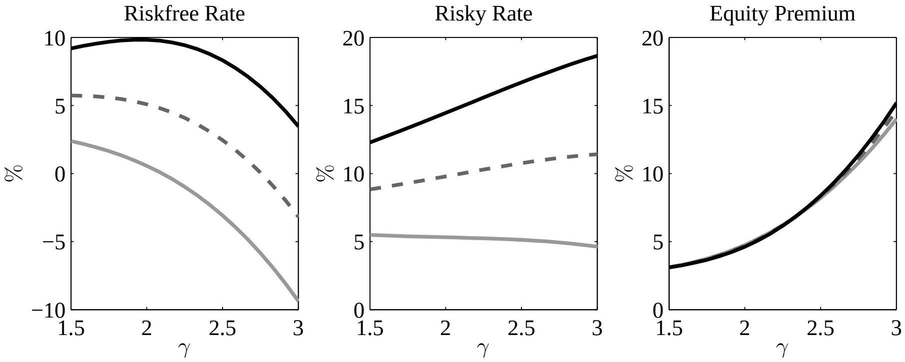


Low $\widehat{x}_t$ ($-\sigma_{\widehat{x}}$), Average value of $\widehat{x}_t$, High value of $\widehat{x}_t$ ($\sigma_{\widehat{x}}$)

Note: Obtained for $\gamma = 2.50$ and $\rho = 0.85$.

Results

Comparative statics: Variations in γ



Low $\widehat{x}_t$ ($-\sigma_{\widehat{x}}$), Average value of $\widehat{x}_t$, High value of $\widehat{x}_t$ ($\sigma_{\widehat{x}}$)

Note: Obtained for $\alpha = 8.6$ and $\rho = 0.85$.

- Note that this was derived with the single–ρ model,
- Results for the 2–ρ model do not differ much See Figure
- What does it buy us?

- Equity premium is thought to be countercyclical
- Correlation between $E_t(r_{t+1} - r_t^f)$ and HP–filtered log–consumption

	Single ρ		2–ρ	
γ	α	Div.	α	Div.
1.00	34.681	0.34	39.050	-0.40
1.50	22.005	0.20	26.900	-0.44
2.00	14.198	-0.02	17.900	-0.46
2.50	8.606	-0.06	11.050	-0.46
3.00	4.865	-0.06	6.350	-0.47

▸ Consumption Claims

- More pronounced counter–cyclicality because of variable ambiguity

▸ See Figure

- An Ambiguity Aversion Premium
- The certainty equivalent assuming all the uncertainty is pure risk ("rational expectations"):

$$CE(0) \equiv u^{-1}\left(\int u\left(C\exp\left(g\right)\right)dH\left(g;x\right)\right)$$

where $H\left(g;x\right) \equiv \mathcal{N}\left(\hat{x}+\bar{g},\sigma_x^2+\sigma_g^2\right)$

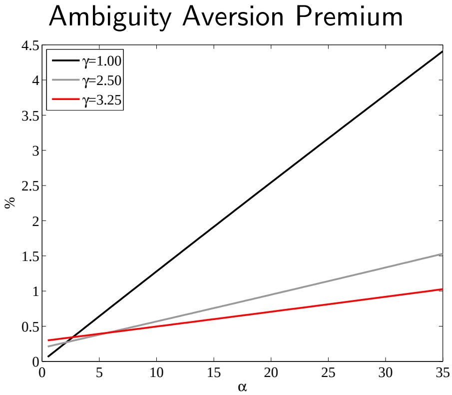
- Certainty equivalent:

$$CE(\alpha) \equiv u^{-1}\left(\phi^{-1}\left(\int \phi\left(\int u\left(C\exp\left(g\right)\right)dF\left(g;x\right)\right)dF\left(x\right)\right)\right)$$

where $F\left(g;x\right) \equiv \mathcal{N}\left(x+\bar{g},\sigma_x^2+\sigma_g^2\right)$ and $F\left(x\right) \equiv \mathcal{N}\left(\hat{x},\frac{\widehat{P}}{1-\rho^2}\right)$.

- Define the premium as

$$\frac{\mathscr{A}}{100} = \frac{CE(0)-CE(\alpha)}{\int C\exp\left(g\right)dH\left(g;x\right)}$$



- Camerer (1999) reports experimental data (several experiments).
- Suggests $\mathscr{A}$ to be below 10%.

Concluding Remarks

- Analyze the quantitative effects of ambiguity aversion on asset price equilibria
- Apply the KMM framework to a Lucas' (1978) tree model
- Provides a good framework to understand the equity premium
- Accounts for most of the puzzle

Appendix
Supplementary Material

- Updating equation

$$\widehat{x}_{t+1} = \rho\widehat{x}_t + K\nu_{t+1}$$

- The agent uses all observed variables, such that

$$\nu_{t+1} = \begin{pmatrix} g_{c,t+1} - \overline{g}_c - \rho\widehat{x}_t \\ g_{d,t+1} - \overline{g}_d - \psi\rho\widehat{x}_t \end{pmatrix} = \begin{pmatrix} \rho(x_t - \widehat{x}_t) + \sigma_c\varepsilon_{c,t+1} + \sigma_x\varepsilon_{x,t+1} \\ \psi\rho(x_t - \widehat{x}_t) + \sigma_d\varepsilon_{d,t+1} + \psi\sigma_x\varepsilon_{x,t+1} \end{pmatrix}$$

- The kalman gain matrix, K, is given by

$$K = \rho P(1\ \psi)F^{-1} \text{ with } F = \begin{pmatrix} P + \sigma_c^2 & \psi P \\ \psi P & \psi^2 P + \sigma_d^2 \end{pmatrix}$$

and P solves

$$P = \rho^2 P - P(1\ \psi)F^{-1}(1\ \psi)'P' + \sigma_x^2$$

Go Back

- Updating equations

$$\widehat{x}^{(i)}_{j,t+1} = \rho_j \widehat{x}_{j,t} + K_j \nu^{(i)}_{j,t+1}$$

where

$$\nu^{(i)}_{j,t+1} = \begin{pmatrix} g_{c,i,t+1} - \overline{g}_{c,j} - \rho_j \widehat{x}_{j,t} \\ g_{d,i,t+1} - \overline{g}_{d,j} - \psi_j \rho_j \widehat{x}_{j,t} \end{pmatrix}$$

with $i, j = \{\ell, h\}$.

- The kalman gain matrix, K, is given by

$$K_j = \rho_j P_j (1 \ \psi_j) F_j^{-1} \text{ with } F_j = \begin{pmatrix} P_j + \sigma^2_{c,j} & \psi_j P_j \\ \psi_j P_j & \psi_j^2 P_j + \sigma^2_{d,j} \end{pmatrix}$$

and P_j solves

$$P_j = \rho_j^2 P_j - P_j (1 \ \psi_j) F_j^{-1} (1 \ \psi_j)' P_j' + \sigma^2_{x,j}$$

with $j = \{\ell, h\}$.

Go Back

The prior probability assigned to the low persistence model, η_t is updated as

$$\eta_{t+1}^{(i)} = \frac{\eta_t \mathscr{L}(\nu_{\ell,t+1}^{(i)}, F_\ell)}{\eta_t \mathscr{L}(\nu_{\ell,t+1}^{(i)}, F_\ell) + (1-\eta_t)\mathscr{L}(\nu_{h,t+1}^{(i)}, F_h)}$$

where

$$\mathscr{L}(\nu_{j,t+1}^{(i)}, F_j) = \frac{1}{2\pi|F_j|}\exp\left(-\frac{\nu_{j,t+1}^{(i)\prime} F_j^{-1} \nu_{j,t+1}^{(i)}}{2}\right)$$

with $i, j = \{\ell, h\}$.

Go Back

$$V(C_t, x_t^h, x_t^\ell, \eta_t) = (1-\beta)u(C_t) + \beta\Phi^{-1}(\mathscr{V}_{t+1})$$

with

$$\mathscr{V}_{t+1} \equiv \eta_t E_{\mu_{\ell,t}}\left[\Phi\left(E_{\pi_{x_{\ell,t}}}\left[V\left(C_{t+1}^{(\ell)}, x_{h,t+1}^{(\ell)}, x_{\ell,t+1}^{(\ell)}, \eta_{t+1}^{(\ell)}\right)\right]\right)\right] + (1-\eta_t)E_{\mu_{h,t}}\left[\Phi\left(E_{\pi_{x_{h,t}}}\left[V\left(C_{t+1}^{(h)}, x_{h,t+1}^{(h)}, x_{\ell,t+1}^{(h)}, \eta_{t+1}^{(h)}\right)\right]\right)\right]$$

Go Back

Leverage	$\psi_\ell = \psi_h$	3	3
Constant g_c	$\overline{g}_{c,\ell} = \overline{g}_{c,h}$	0.019256	0.019256
Constant g_d	$\overline{g}_{d,\ell} = \overline{g}_{d,h}$	0.006241	0.006241
Persistence	ρ_ℓ	0.30	0.30
Volatility ε_c	σ_{c_ℓ}	0.020578	0.020578
Volatility ε_d	σ_{d_ℓ}	0.097742	0.097742
Volatility ε_x	σ_{x_ℓ}	0.025028	0.025028
Persistence	ρ_h	0.85	0.90
Volatility ε_c	σ_{c_h}	0.014864	0.013289
Volatility ε_d	σ_{d_h}	0.105598	0.106400
Volatility ε_x	σ_{x_h}	0.024657	0.025763

Go Back

	Consumption		Dividend	
	$E(g)$	$\sigma(g)$	$E(g)$	$\sigma(g)$
Bayesian	0.0193	0.0348	0.0064	0.1348
Twisted	0.0003	0.0385	-0.0505	0.1434
	$Sk(g)$	$Ku(g)$	$Sk(g)$	$Ku(g)$
Bayesian	-0.0001	0.0283	-0.0003	0.0627
Twisted	-0.1145	-0.0340	-0.1373	0.0159

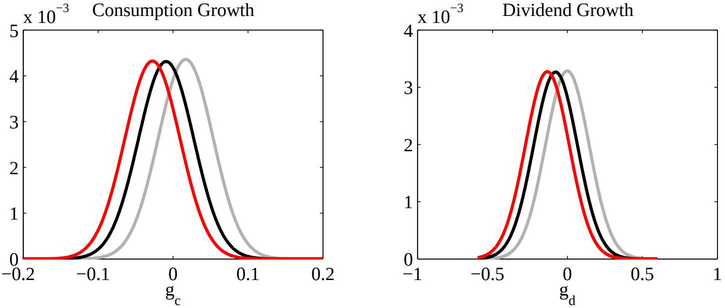
Go Back

γ	α	$E(r)$	$E(r-r^f)$	$\sigma(r^f)$	$\sigma(r)$	$\sigma(r-r^f)$
Data		9.83	7.38	2.63	13.94	13.66
Bayes.	r^f =10.03	10.63	0.60	2.40	6.50	5.95
1.00	34.681	5.70	3.26	0.85	2.45	2.24
1.50	22.005	6.50	4.06	1.37	3.77	3.45
2.00	14.198	7.33	4.89	1.93	5.12	4.68
2.50	8.606	8.17	5.73	2.49	6.49	5.94
3.00	4.865	9.04	6.60	3.03	7.88	7.21
$\rho = 0.90$	5.612	8.81	6.37	2.78	7.56	6.93
2–ρ	11.050	8.40	5.96	3.68	5.92	6.08

Go Back

Appendix

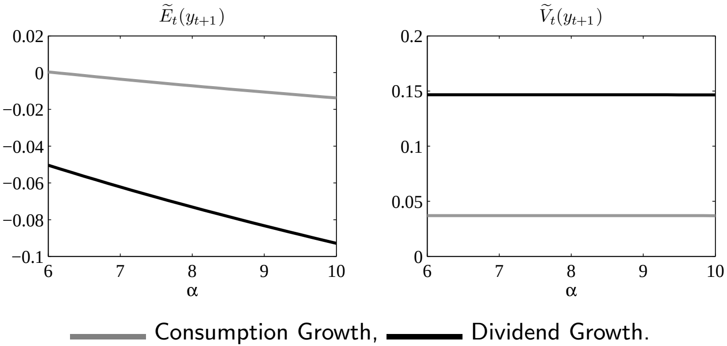
Shifts in the Twisted Distribution (α)



$\alpha = 1$, $\alpha = \widehat{\alpha}$, $\alpha = 15$.

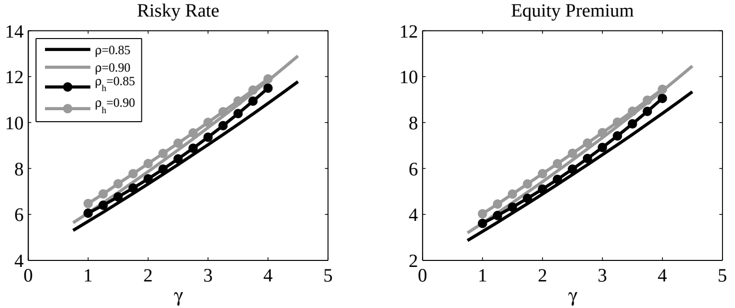
Appendix

Twisted Conditional Moments as a function of α



Go Back

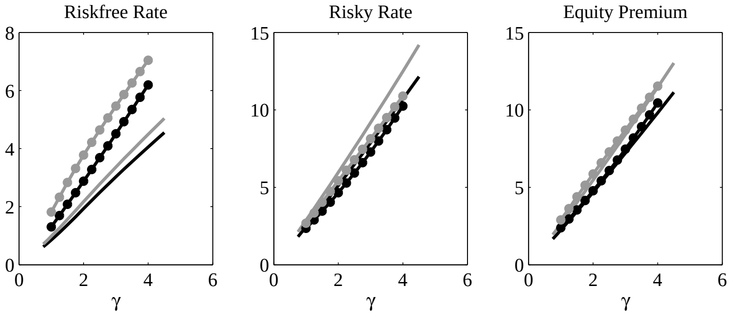
Consumption Claims – First Order Moments



Note: In each case, α was set such that $E(r^f) = 2.44\%$.

Go Back

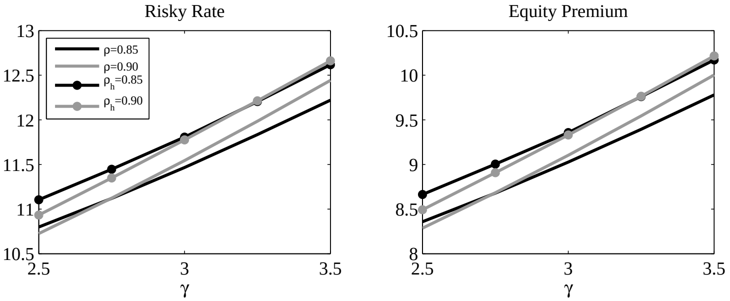
Consumption Claims – Second Order Moments



Note: In each case, α was set such that $E(r^f) = 2.44\%$.

Go Back

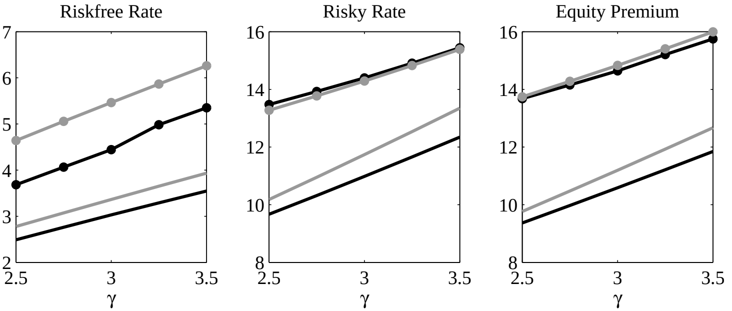
Dividend Claims – First Order Moments



Note: In each case, α was set such that $E(r^f) = 2.44\%$.

Go Back

Dividend Claims – Second Order Moments



Note: In each case, α was set such that $E(r^f) = 2.44\%$.

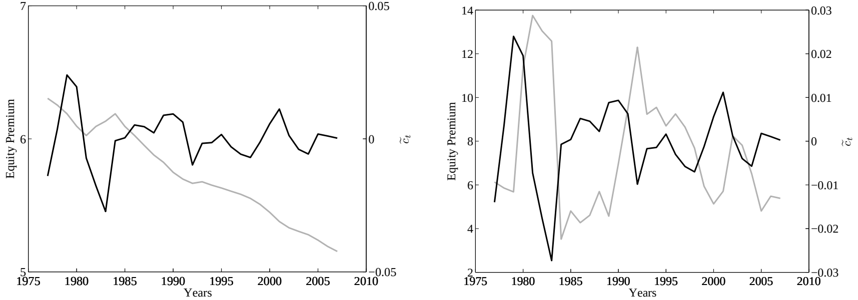
Go Back

	Single ρ		2–ρ	
γ	α	Cons.	α	Cons.
1.00	34.681	0.37	39.050	-0.40
1.50	22.005	0.21	26.900	-0.44
2.00	14.198	0.00	17.900	-0.46
2.50	8.606	-0.04	11.050	-0.46
3.00	4.865	-0.05	6.350	-0.47

Go Back

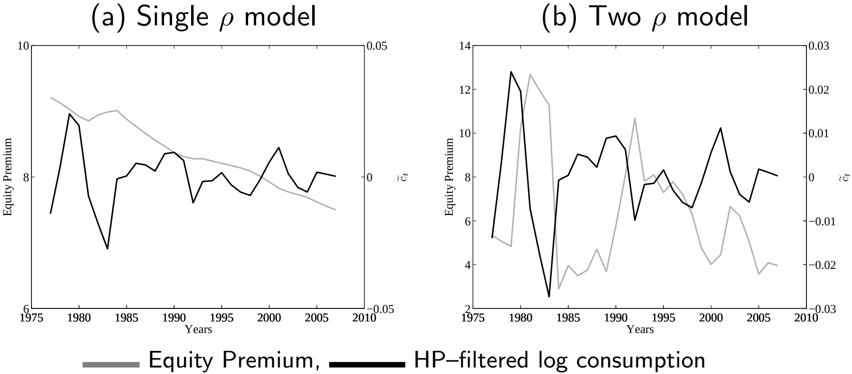
Consumption Claims

(a) Single ρ model | (b) Two ρ model



Equity Premium, HP–filtered log consumption

Dividend Claims



Go Back