Imperfect information and the business cycle *

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Abstract

We estimate a small, rational expectations, new Keynesian, model where the agents face a signal extraction problem. Learning about the state of the economy adds an important endogenous propagation mechanism that improves the dynamic behavior of the model. Not only can the model generate inflation inertia but it also performs better –based on likelihood criteria– than other existing versions of the NK model which feature alternative inertial mechanisms such as real rigidities and backward looking agents.

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Introduction

The baseline New Keynesian (NK) model has well known empirical flaws, in particular regarding macroeconomic dynamics. In order to deal with its weaknesses, extended versions of the NK model contain a number of additional features (see Rotemberg and Woodford, 1997, Gali and Gertler, 1999, Christiano et al., 2005, Altig et al., 2005, and Smets and Wouters, 2005). The most prominent of these features are various types of real rigidities, such as habit persistence, capital or investment adjustment costs, capital utilization, predetermined expenditure and so on. And backward looking price setting schemes for subsets of the agents. Collard and Dellas, 2005, show that the latter feature plays a key role for the ability of the NK model to exhibit inflation inertia, as captured by the hump-shaped response of inflation to monetary policy shocks.

Both real rigidities and backward inflation indexation schemes have their critics[1]. Consequently, considerable effort has been expanded during the last few years in the development of alternative propagation-inertia mechanisms that may be less controversial. Sticky information (Mankiw and Reis, 2002) and signal extraction (Collard and Dellas, 2005) are two prominent examples of this approach. Collard and Dellas, 2005, have showed that a *calibrated* NK model with monetary mis-perceptions and signal extraction a la Lucas and real rigidities has good dynamic properties and also overall performance.

The objective of this paper is to carry out an econometric evaluation of the role played by the various inertial mechanisms as well the properties they induce onto the NK model. In particular, we estimate and compare the performance of the NK model under three alternative specifications: a) The original, baseline version, which does not contain any real rigidities. b) The version with real rigidities and non forward looking agents. And c) a version without any real rigidities but with rational, forward looking agents who solve a signal extraction problem. The signal extraction problem arises from measurement errors in reported data[2].

The models are estimated on US data over the 1965-2000 period and are compared in terms of two criteria: The log-likelihood. And the properties of the impulse response function of inflation to a monetary policy shock. Christiano et al., 2005, and Mankiw and Reis, 2002, have argued that the ability of a monetary model to generate a hump shape IRF is the litmus test for this class of models.

[1]For instance, price indexation seems to be at variance with the empirical evidence regarding pricing behavior as documented by a recent ECB report (Dhyne et al. 2005). Namely, the observation that individual price changes do not move in tandem with aggregate inflation. Similarly, adjustment costs are often criticized as representing an ad hoc feature.

[2]Collard and Dellas, 2005, show that such "noise" in preliminary aggregate monetary data plays an important role in the monetary transmission mechanism by establishing two things. First, that the measurement error –the difference between preliminary and revised data– is quantitatively important. And second, that this error represents unperceived money and matters significantly for economic activity.

These two aspects, relative performance and inflation dynamics, are among the key differences between this paper and other work in the literature that tests the empirical validity of the NK model under alternative specifications. For instance, Eichenbaum and Fisher, 2004, find that an estimated version of the NK model with backward indexation is consistent with the data (as judged by the J-statistic in the context of GMM estimation). de Walque, Smets and Wouters, 2004, find that the Smets and Wouters model performs well even when the parameter of backward indexation is close to zero. But the fact that the model without indexation is not rejected by the data does not mean that it performs satisfactorily along the *inflation dynamics* dimension. Moreover, the fact that a model is not rejected by the data does not mean that it is the best model within a particular class. These issues will be further highlighted below[3].

We find that the specification with the signal extraction problem and no real rigidities not only has good dynamic properties (e.g. inflation inertia) but it also outperforms the other versions according to standard likelihood criteria. Furthermore, the estimated parameters have plausible values that agree with those typically estimated in the literature, and the amount and location of noise is plausible. The model with real rigidities comes a distant second. As in de Walque, Smets and Wouters, 2004, that version's performance is not adversely affected by the fact that the estimated parameter of backward indexation is close to zero[4]. The baseline version of the NK model lags far behind the rest. We interpret these findings as suggesting that neither real rigidities nor backward looking agents are needed in order for the sticky price model to be a successful monetary model of the business cycle.

The remaining of the paper is organized as follows. Section 1 presents the model. Section 2 discusses the econometric methodology. Section 3 presents the main results. The last section offers some concluding remarks.

[3]Another related paper is Lippi and Neri, 2007. Lippi and Neri estimate a small Keynesian model under signal extraction but, having a different objective, they do not examine the stochastic properties of their model. Moreover, they do not compare the performance of their model to that of alternative NK specifications, so one cannot judge its *relative* success.

[4]This is true in terms of the likelihood value. But it remains true that in the absence of a significant value for the parameter of backward indexation the model still fails to exhibit a hump shaped response to a monetary shock.

1 The Model

1.1 The Household

There exists an infinite number of households distributed over the unit interval and indexed by $j \in [0,1]$. The preferences of household j are given by

$$\mathbb{E}_t \sum_{\tau=0}^{\infty} \beta^{\tau} \chi_{t+\tau} \left[\log(c_{t+\tau} - x_{t+\tau}) + \nu^h (1 - h_{t+\tau}) \right] \tag{1}$$

where $0 < \beta < 1$ is a constant discount factor, c_t denotes consumption in period t, and h_{jt} is the quantity of labor supplied by the representative household of type j. χ_t is a preference shock that is assumed to follow an AR(1) process of the form

$$\log(\chi_t) = \rho_{\chi} \log(\chi_{t-1}) + (1 - \rho_{\chi}) \log(\overline{\chi}) + \varepsilon_{\chi,t}$$

where $|\rho_{\chi}| < 1$ and $\varepsilon_{\chi,t} \rightsquigarrow \mathscr{N}(0, \eta_{\chi}^2))$.

x_t denotes an external habit stock which is assumed to be proportional to past aggregate consumption:

$$x_t = \vartheta \overline{c}_{t-1} \text{ with } \vartheta \in (0,1).$$

In each period, household j faces the budget constraint

$$B_t + P_t c_t = R_{t-1} B_{t-1} + W_t h_t + \Pi_t \tag{2}$$

where B_t is nominal bonds. P_t, the nominal price of goods. c_t denotes consumption expenditures. W_t is the nominal wage. Ω_t is a nominal lump-sum transfer received from the monetary authority and Π_t denotes the profits distributed to the household by the firms.

This yields the following set of first order conditions

$$\chi_t (c_t - \theta \overline{c}_{t-1})^{-1} = \lambda_t \tag{3}$$

$$\nu_h \chi_t = \lambda_t w_t \tag{4}$$

$$\lambda_t = \beta R_t E_t \frac{\lambda_{t+1}}{\pi_{t+1}} \tag{5}$$

1.2 The firms

1.2.1 Final Good Producers

The final good, y is produced by combining intermediate goods, y_i, by perfectly competitive firms. The production function is given by

$$y_t = \left(\int_0^1 y_{it}^{\theta} \mathrm{d}i \right)^{\frac{1}{\theta}} \tag{6}$$

where $\theta \in (-\infty, 1)$. Profit maximization and free entry lead to the general price index

$$P_t = \left(\int_0^1 P_{it}^{\frac{\theta}{\theta-1}} \mathrm{d}i\right)^{\frac{\theta-1}{\theta}} \tag{7}$$

Profit maximization gives rise to the following demand function for good i

$$y_{it} = \left(\frac{P_{it}}{P_t}\right)^{\frac{1}{\theta-1}} y_t \tag{8}$$

The final good may be used for consumption.

1.2.2 Intermediate goods producers

Each firm i, $i \in (0, 1)$, produces an intermediate good by means of labor according to a constant returns–to–scale technology, represented by the Cobb–Douglas production function

$$y_{it} = a_t h_{it} \tag{9}$$

where h_{it} denotes the labor input used by firm i in the production process. a_t is an exogenous technology shock which is assumed to follow an AR(1) process of the form

$$\log(a_t) = \rho_a \log(a_{t-1}) + (1 - \rho_a) \log(\overline{a}) + \varepsilon_{a,t}$$

where $|\rho_a| < 1$ and $\varepsilon_{a,t} \rightsquigarrow \mathscr{N}(0, \eta_a^2)$.

Intermediate goods producers are monopolistically competitive, and therefore set prices for the good they produce. We follow Calvo in assuming that firms set their prices for a stochastic number of periods. In each and every period, a firm either gets the chance to adjust its price (an event occurring with probability $(1 - \xi)$) or it does not. If it does not get the chance, then it is assumed to set prices according to

$$P_{it} = \pi_{t-1}^{\gamma} \overline{\pi}^{1-\gamma} P_{it-1} \tag{10}$$

where $\gamma \in (0, 1)$ determines the price indexation scheme. Note that setting $\gamma = 1$ we retrieve the lagged indexation specification used by **?**, while when $\gamma = 0$ prices are indexed on steady state inflation.

On the other hand, a firm i that sets its price optimally in period t chooses a price, $P_t^{\star}$, in order to maximize:

$$\max_{P_t^{\star}} \mathbb{E}_t \sum_{\tau=0}^{\infty} \Phi_{t+\tau} (1 - \gamma)^{\tau} \left(P_t^{\star} \Xi_{t,\tau} - P_{t+\tau} s_{t+\tau}\right) y_{it+\tau}$$

subject to the total demand (8) and

$$\Xi_{t,\tau} = \begin{cases} \pi_t^{\gamma} \overline{\pi}^{1-\gamma} \times \ldots \times \pi_{t+\tau-1}^{\gamma} \overline{\pi}^{1-\gamma} & \text{if } \tau \geqslant 1 \\ 1 & \text{otherwise} \end{cases}$$

Note that we have $\Xi_{t,\tau+1} = \pi_t^{\gamma}\overline{\pi}^{1-\gamma}\Xi_{t+1,\tau}$.

$\Phi_{t+\tau}$ is an appropriate discount factor derived from the household's evaluation of future relative to current consumption. This leads to the price setting equation

$$P_t^{\star} = \frac{1}{\theta}\frac{\mathbb{E}_t\sum_{\tau=0}^{\infty}(1-\xi)^{\tau}\Phi_{t+\tau}P_{t+\tau}^{\frac{\theta-2}{\theta-1}}\Xi_{t,\tau}^{\frac{1}{\theta-1}}s_{t+\tau}y_{t+\tau}}{\mathbb{E}_t\sum_{\tau=0}^{\infty}(1-\xi)^{\tau}\Phi_{t+\tau}\Xi_{t,\tau}^{\frac{\theta}{\theta-1}}P_{t+\tau}^{\frac{1}{1-\theta}}y_{t+\tau}} \tag{11}$$

Since the price setting scheme is independent of any firm specific characteristic, all firms that reset their prices will choose the same price.

In each period, a fraction ξ of contracts ends and $(1-\xi)$ survives. Hence, from (7) and the price mechanism, the aggregate intermediate price index writes

$$P_t = \left(\xi P_t^{\star\frac{\theta}{\theta-1}} + (1-\xi)(\pi_{t-1}^{\gamma}\overline{\pi}^{1-\gamma}P_{t-1})^{\frac{\theta}{\theta-1}}\right)^{\frac{\theta-1}{\theta}} \tag{12}$$

1.3 Monetary Policy

The monetary policy is assumed to take the form

$$\log(R_t) = \rho_r\log(R_{t-1}) + (1-\rho_r)(\kappa_y(\log(y_t)-\log(\overline{y}_t)) + \kappa_{\pi}(\log(\pi_t)-\log(\overline{\pi})) + \epsilon_m$$

where $\overline{\pi}$ represents the level of steady state inflation. $\overline{y}_t$ is potential (that is, flexible price) output and ϵ_m is a policy shock.

with $|\rho_y| < 1$ and $\varepsilon_{\overline{y},t} \rightsquigarrow \mathscr{N}(0,\eta_y^2)$.

1.4 Information

As stated in the introduction, our objective is compare the performance of the NK model under the standard specification (with real rigidities andor price indexation) to that without real rigidities and agents who solve a signal extraction problem. The source of the signal extraction problem is measurement error in some of the aggregate variables. The existence of significant measurement error in macroeconomic variables is well known. Its size can be documented by examining the real data times series at the Philadelphia FED. For instance, using this database, Collard and Dellas, 2006, establish that the noise in preliminary data on monetary aggregates –the difference between preliminary and finally revised data– is quantitative substantial. Moreover, they also show that this measurement error plays an important role in the business cycle.

For mis–measured variable x_i we assume that

$$x_{it}^{\star} = x_{it}^{\mathrm{T}} + \eta_{it}$$

where x_t^{T} denotes the true value of the variable and η_t is a noisy process that satisfies $E(\eta_t) = 0$ for all t; $E(\eta_t \varepsilon_{a,t}) = E(\eta_t \varepsilon_{g,t}) = 0$; and

$$E(\eta_t \eta_k) = \begin{cases} \sigma_\eta^2 & \text{if } t = k \\ 0 & \text{Otherwise} \end{cases}$$

The agents are assumed to learn about the true aggregate state of the economy gradually using the Kalman filter, based on a set of signals on aggregate variables. We offer a more detailed discussion of the modelling of the measurement errors as well as the solution method in the appendix.

Before proceeding, it is necessary to make two points. First, knowledge of the true *aggregate* state of the economy matters for the agents because individual price setting depends on expectations of future nominal marginal cost and marginal revenue, which in turn depend on future aggregate prices, wages and so on.

And second, for the informational considerations emphasized in this paper to be taken seriously, it is essential that the informational constraints be sensible. We assume that the nominal interest rate is perfectly observable while the observations of the output gap and the inflation rate are ridden with measurement errors. The agents are given noisy signals on the output gap $(y_t - \overline{y}_t)$ and inflation (π_t). The estimated policy rule takes the form

$$\log(R_t) = \rho_r \log(R_{t-1}) + (1 - \rho_r)(\kappa_y(\log(y_t) - \log(\overline{y}_{t|t})) + \kappa_\pi(\log(\pi_t) - \log(\overline{\pi})) + \epsilon_m$$

where $\overline{y}_{t|t}$ is perceived potential output.

1.5 Equilibrium

In equilibrium, we have $y_t = c_t$. We estimate a log–linear version of the model which has the following IS–PC representation[5]

$$\widehat{y}_t = \frac{\vartheta}{1+\vartheta}\widehat{y}_{t-1} + \frac{1}{1+\vartheta}\mathbb{E}_t\widehat{y}_{t+1} - \frac{1-\vartheta}{1+\vartheta}(\widehat{R}_t - \mathbb{E}_t\widehat{\pi}_{t+1}) + \frac{1-\vartheta}{1+\vartheta}(\widehat{\chi}_t - \mathbb{E}_t\widehat{\chi}_{t+1}) \tag{13}$$

$$\begin{aligned}\widehat{\pi}_t =& \frac{\gamma}{1+\beta\gamma}\widehat{\pi}_{t-1} + \frac{\beta}{1+\beta\gamma}\mathbb{E}_t\widehat{\pi}_{t+1} + \frac{\xi(1-\beta(1-\xi))}{(1-\xi)(1+\beta\gamma)}\left(\frac{1}{1-\vartheta}\right)\widehat{y}_t \\ &- \frac{\xi(1-\beta(1-\xi))}{(1-\xi)(1+\beta\gamma)}\frac{\vartheta}{1-\vartheta}\widehat{y}_{t-1} - \frac{\xi(1-\beta(1-\xi))}{(1-\xi)(1+\beta\gamma)}\widehat{a}_t\end{aligned} \tag{14}$$

$$\widehat{R}_t = \rho_r\widehat{R}_{t-1} + (1-\rho_r)(\kappa_y(\widehat{y}_t - \widehat{\overline{y}}_t) + \kappa_\pi\widehat{\pi}_t) + \epsilon_m \tag{15}$$

Note that setting both γ and ϑ to zero implies a NK model with only nominal rigidities. Setting $\vartheta > 0$ and $\gamma = 0$, is the NK model with real frictions but no backward price indexation. Finally, setting both $\vartheta > 0$ and $\gamma > 0$ results in a model with both real frictions and lagged indexation (similar to Christiano et al, 2005).

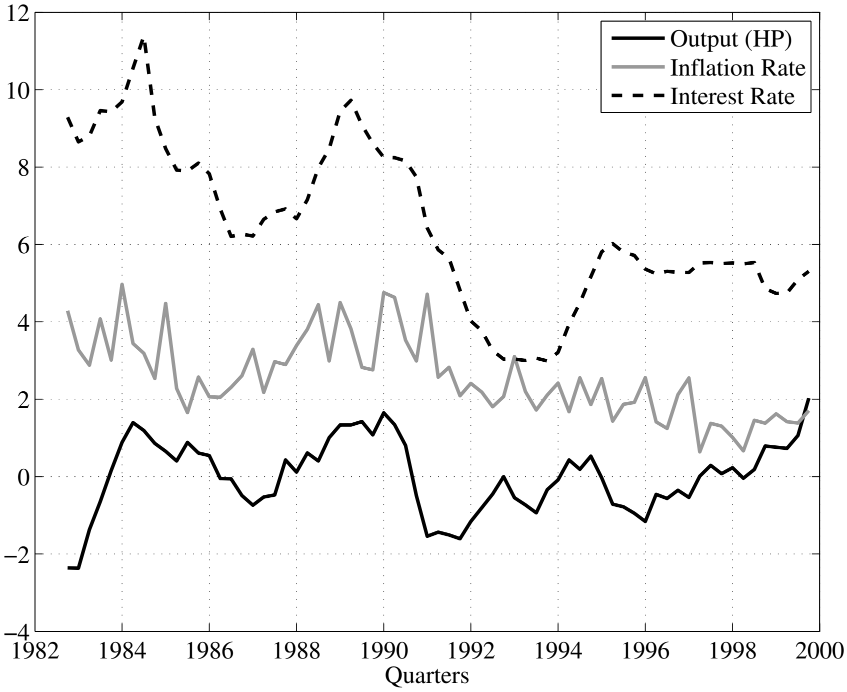
[5] The details of the derivation are given in appendix **??**.

2 Econometric Methodology

2.1 Data

The model is estimated on US quarterly data for the sample period 1966:IV–1999:IV. The data are taken from the Federal Reserve Economic Database. Output is measured by real GDP, inflation is the annualized quarterly change in the GDP deflator, while the nominal interest rate is the Federal Funds Rate. The output series is detrended using the H-P filter.

Figure 1: US Data



2.2 Estimation method

We do not estimate all the parameters of the model as some of them cannot be identified in the steady state and do not enter the log–linear representation of the economy. This is the case for the demand elasticity, θ, and the weight on leisure, ν_h, in the utility function. The discount factor, β, is set to 0.9926, which implies an annual discount rate of 3%. We estimate the vector of parameters $\Psi = \{\vartheta, \xi, \gamma, \rho_a, \rho_\chi, \rho_y, \sigma_a, \sigma_\chi, \sigma_y, \sigma_1, \sigma_2, \rho_r, \kappa_\pi, \kappa_y\}$. Ψ is estimated relying on a Bayesian maximum likelihood procedure. As a first step of the procedure, the log–linear system (13)–(15) is solved using the Blanchard-Khan method. In the specification with the signal extraction, the model is solved according to the method outlined in the Appendix. The Kalman filter is then used on the solution of the model to form the log–likelihood, $\mathcal{L}_m(\{\mathscr{Y}_t\}_{t=1}^T, \Psi)$, of each model, m. Once the posterior mode is obtained by maximizing the likelihood function, we obtain the posterior density function using the Metropolis–Hastings algorithm (see Lubic and Schorfheide, 2004).

Table 1 presents the prior distribution of the parameters. The habit persistence parameter, ϑ, is beta distributed as it is restricted to belong to the [0,1) interval. The average of the distribution is set to 0.70, which is in line with the prior distribution used by Smets and Wouters, 2003. The steady state inflation rate and real interest rate have a Γ-distribution with means 4% and 2% per year respectively.

Table 1: Priors

Parameter	Range	Type	Mean	St. error	95% Conf. Int.
		Preferences			
ϑ	[0,1)	Uniform	0.50	0.29	[0.03;0.97]
		Nominal Rigidities			
ξ	[0,1]	Uniform	0.50	0.29	[0.03,0.97]
γ	[0,1)	Uniform	0.50	0.29	[0.03;0.97]
		Taylor Rule			
ρ_r	[0,1)	Beta	0.80	0.10	[0.57;0.95]
κ_π	$\mathbb{R}_+$	Gamma	1.50	0.20	[1.13;1.91]
κ_y	$\mathbb{R}_+$	Gamma	0.25	0.10	[0.09;0.48]
		Forcing Variables			
ρ_a	[0,1)	Beta	0.85	0.10	[0.61,0.98]
ρ_χ	[0,1)	Beta	0.85	0.10	[0.61,0.98]
ρ_y	[0,1)	Beta	0.85	0.10	[0.61,0.98]
σ_a	$\mathbb{R}_+$	Inv. Gamma	0.40	0.35	[0.19,0.92]
σ_χ	$\mathbb{R}_+$	Inv. Gamma	0.20	0.17	[0.09,0.46]
σ_y	$\mathbb{R}_+$	Inv. Gamma	0.40	0.35	[0.19,0.92]
		Noise			
σ_p	$\mathbb{R}_+$	Inv. Gamma	0.50	0.26	[0.24;1.14]
σ_y	$\mathbb{R}_+$	Inv. Gamma	0.50	0.26	[0.24;1.14]

Note: The Inverse Gamma priors take the form $p(\sigma|\nu,s) \propto \sigma^{-(1+\nu)}e^{\nu s^2/2\sigma^2}$. We set $s = 0.40$ and $\nu = 4$.

The parameters pertaining to the nominal rigidities are distributed according to a beta distribution as they belong to the [0,1) interval. The average probability of price resetting is set to 0.25, implying that a firm expects to reset prices on average every four quarters. Following Smets and Wouters, 2003, the average lagged price indexation parameter, γ, is set to 0.75.

The persistence parameter of the Taylor rule, ρ_r, has a beta prior over [0,1) so as to guarantee the stationarity of the rule. The prior distribution is centered on 0.8, a value in line with existing estimations of this parameter (Lubic and Schorfheide, 2004). The reaction to inflation, κ_π, and output, κ_y, is assumed to be positive, and with a gamma distribution centered on 1.5 and 0.25 respectively. These values correspond to those commonly estimated in the literature.

We have little knowledge of the processes that describe the forcing variables. We assume a beta distribution for the persistence parameter in order to guarantee the stationarity of the process. Each distribution is centered on 0.85. Volatility is assumed to follow an inverse gamma distribution (to guarantee positiveness), centered on 0.5%. However, in order to take into account the limited knowledge we have regarding these process we impose non informative priors. The same strategy is applied for the noise process in the signal extraction model.

3 Estimation Results

Table 2 reports the posterior estimates for the three versions. Several observations are in order. First, the estimated coefficients that are common across the three models do not differ much across the various specifications, with three exceptions. The degree of price flexibility; the policy response to the output gap; and the volatility of the preference shock. Nonetheless, all parameter values are well within the range of reported values in the literature.

Second, the model with real rigidities does not need any backward looking agents in order to achieve its maximum likelihood value. That is, the estimated coefficient on backward indexation is virtually zero. This is consistent with the findings of de Walque, Smets and Wouters, 2004.

Third, the estimated variance of the noise on inflation is plausible. It is very similar to the value for the measurement error (preliminary vs final release) that has been computed for inflation using the Philadelphia FED real time database (see Collard and Dellas, 2005). The amount of noise on the output gap appears to be quite large but there is no reference in the literature as to what constitutes a plausible value. We think that such a value (and even higher ones) is consistent with the view that it is virtually impossible to measure potential output in the short to medium run[6].

And forth and most important, the specification with a signal extraction clearly dominates the other versions, as can be judged by the large differences in the likelihood.

Judging comparative performance along the dimension of unconditional moments is more ambiguous (Table 3). All three models perform adequately, with the signal extraction specification overestimating the volatility of inflation and underestimating that of output. In our view, the most remarkable feature of this table is that it shows that the model with signal extraction can capture the procyclicality of the nominal interest rate. Canzoneri et al. (2004) have argued

[6]Both the great inflation of the 1970s and the stock price collapse of 2001 have been attributed to the miscalculation of –or uncertainty about – potential output.

Table 2: Posterior Estimates

	Frictionless		Extended version
	P.I.	S.E.	P.I.
ϑ	–	–	0.58
			[0.46,0.77]
ξ	0.14	0.17	0.15
	[0.10,0.19]	[0.13,0.23]	[0.10,0.19]
γ	–	–	0.02
			[0.00,0.11]
ρ_r	0.76	0.78	0.81
	[0.68,0.82]	[0.69,0.84]	[0.73,0.86]
κ_π	1.24	1.18	1.34
	[0.94,1.61]	[0.91,1.51]	[1.04,1.73]
κ_y	0.92	0.82	0.73
	[0.68,1.24]	[0.61,1.12]	[0.50,0.98]
ρ_a	0.53	0.47	0.68
	[0.38,0.69]	[0.35,0.60]	[0.50,0.82]
ρ_χ	0.86	0.86	0.74
	[0.80,0.92]	[0.81,0.91]	[0.64,0.83]
ρ_y	0.69	0.75	0.69
	[0.54,0.81]	[0.60,0.85]	[0.53,0.83]
σ_a	4.55	5.89	3.20
	[1.94,9.32]	[3.08,9.68]	[1.75,5.57]
σ_χ	2.02	2.43	2.52
	[1.52,3.01]	[1.96,3.18]	[1.94,3.53]
σ_y	0.65	0.77	0.89
	[0.52,0.87]	[0.61,1.02]	[0.67,1.31]
σ_1	–	0.40	–
		[0.26,0.73]	
σ_2	–	0.71	–
		[0.36,1.97]	
Likelihood	-290.79	-280.32	-283.28

Note: 95% confidence intervals in brackets. The frictionless version has no real rigidities and no price indexation. The extended version includes them both. S.E is the model with and P.I. is the model without measurement errors.

that there exists no model that does an adequate job in mimicking the cyclical behavior of the nominal interest rate.

We now turn to the dynamics of the models following the shocks. As has been mentioned before, the shape of the IRF of inflation to a monetary policy shock is considered to be a litmus test of the validity of monetary models. It was precisely this consideration that led to the introduction of controversial practices, such as the backward looking pricing schemes, into the NK model. As figures (2)-(4) reveal, the model with signal extraction is the only one that can generate a hump in the response of inflation. The model with real rigidities cannot accomplish this due to the fact that the estimated coefficient of backward indexation is almost zero. This confirms the findings of Collard and Dellas, 2004, about the role of price indexation in models with real rigidities. Note, though, that even if the signal extraction model can indeed produce a hump in inflation and output, the amount of predicted inertia seems to be less than that alleged for the real world. In particular, the effect on impact seems rather substantial relative to that typically reported (see Christiano et al., 2005).

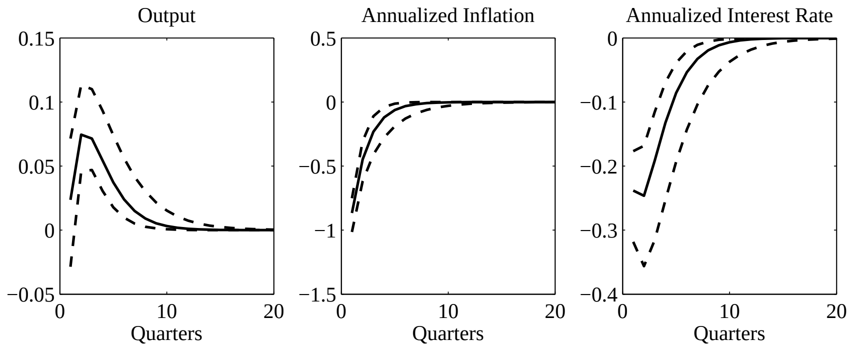
Table 3: Second Order Moments (HP filtered)

	Data	Frictionless		Extended Model
		P.I.	S.E.	P.I.
		Volatilities		
σ_y	0.81	0.70	0.71	0.89
	(0.07)	[0.60,0.82]	[0.62,0.84]	[0.74,1.06]
σ_π	0.73	0.94	0.93	1.09
	(0.05)	[0.80,1.12]	[0.80,1.10]	[0.93,1.34]
σ_R	1.09	1.03	1.03	1.24
	(0.09)	[0.89,1.20]	[0.90,1.18]	[1.05,1.50]
		Correlations		
$\rho(\pi, y)$	0.22	0.19	0.12	0.18
	(0.11)	[0.05,0.36]	[-0.03,0.30]	[0.04,0.32]
$\rho(R, y)$	0.59	0.17	-0.06	0.18
	(0.10)	[0.03,0.31]	[-0.20,0.07]	[0.01,0.35]
		Autocorrelations		
$\rho_y(1)$	0.83	0.55	0.72	0.77
	(0.02)	[0.50,0.59]	[0.70,0.74]	[0.72,0.83]
$\rho_\pi(1)$	0.09	0.41	0.38	0.53
	(0.08)	[0.29,0.53]	[0.27,0.50]	[0.40,0.63]
$\rho_R(1)$	0.89	0.84	0.85	0.87
	(0.02)	[0.81,0.86]	[0.82,0.87]	[0.84,0.90]
$\rho_y(2)$	0.62	0.24	0.40	0.43
	(0.05)	[0.19,0.30]	[0.35,0.44]	[0.35,0.55]
$\rho_\pi(2)$	0.01	0.10	0.09	0.21
	(0.09)	[0.02,0.23]	[0.01,0.19]	[0.08,0.34]
$\rho_R(2)$	0.68	0.58	0.60	0.62
	(0.06)	[0.52,0.63]	[0.54,0.64]	[0.56,0.68]
$\rho_y(4)$	0.06	-0.07	-0.04	-0.08
	(0.15)	[-0.10,-0.03]	[-0.10,0.02]	[-0.13,0.01]
$\rho_\pi(4)$	0.26	-0.12	-0.11	-0.10
	(0.09)	[-0.13,-0.08]	[-0.13,-0.08]	[-0.13,-0.01]
$\rho_R(4)$	0.25	0.11	0.12	0.10
	(0.14)	[0.04,0.17]	[0.06,0.18]	[0.03,0.17]

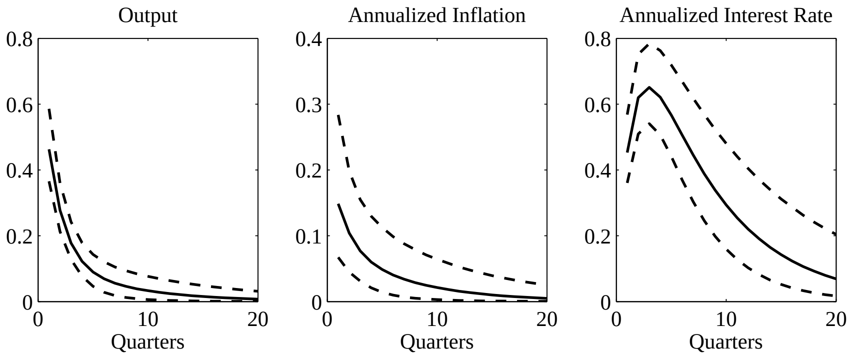
Note: 95% confidence interval into brackets. Frictionless version is the one without any real rigidities and price indexation while the extended version includes them both. S.E is the model with and P.I. is the model without measurement errors.

Figure 2: IRF Frictionless economy, Perfect Information

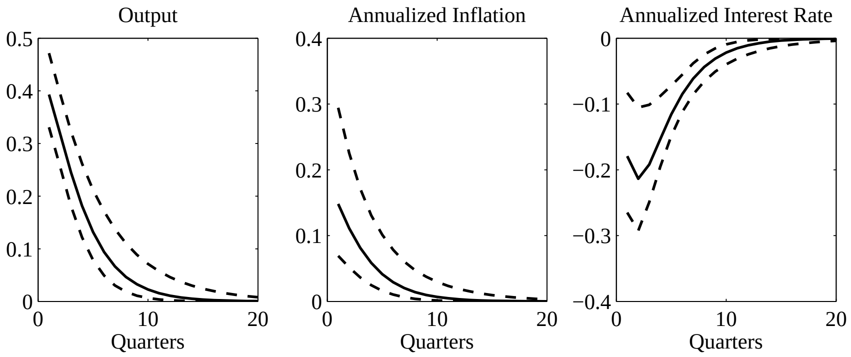
(a) Technology Shocks



(b) Preference Shocks



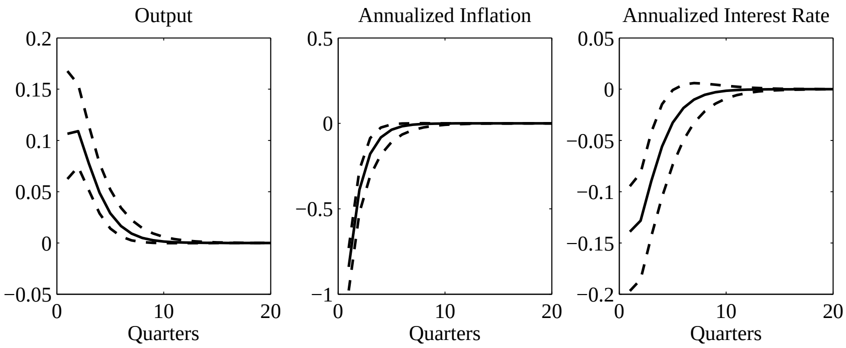
(c) Monetary Policy Shocks



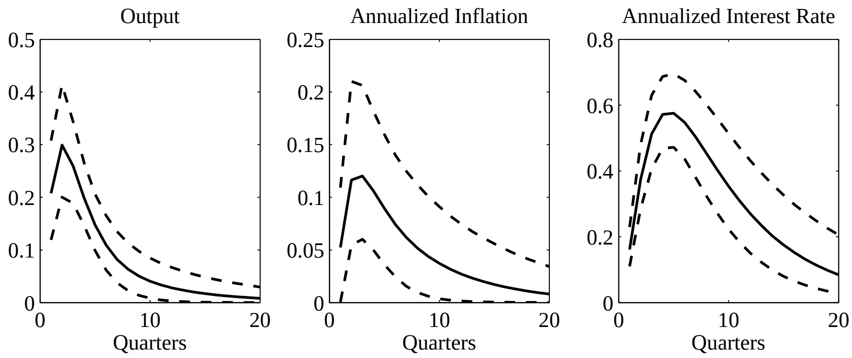
Note: Plain line: median of the distribution, Dashed line: 95% Confidence interval

Figure 3: IRF Frictionless economy, Signal extraction

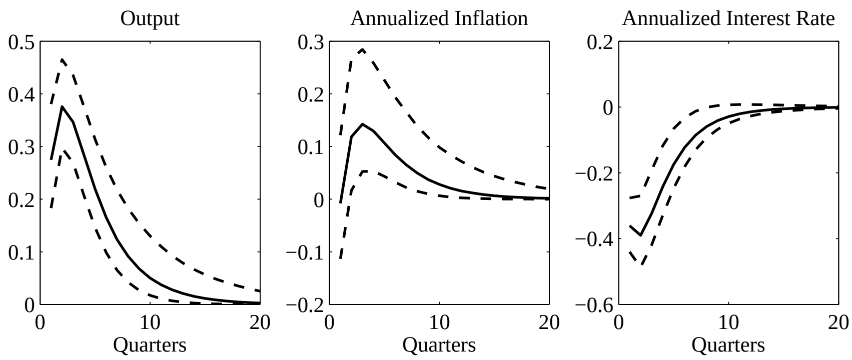
(a) Technology Shocks



(b) Preference Shocks



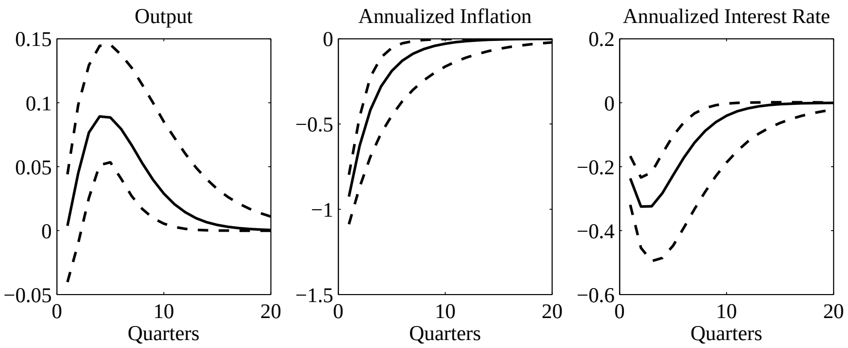
(c) Monetary Policy Shocks



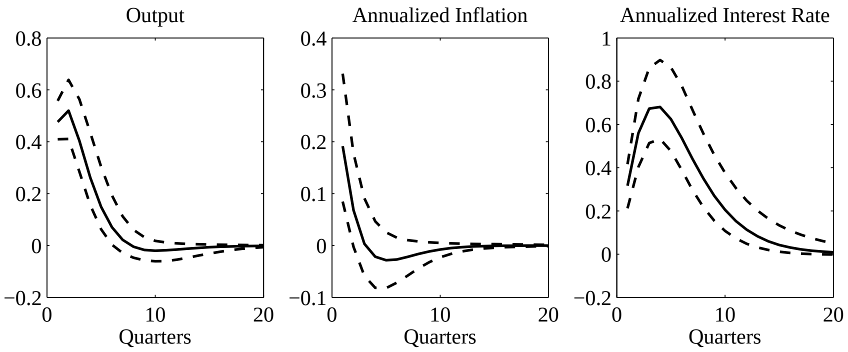
<u>Note:</u> Plain line: median of the distribution, Dashed line: 95% Confidence interval

Figure 4: IRF: Habit Persistence and Price Indexation economy, Perfect Information

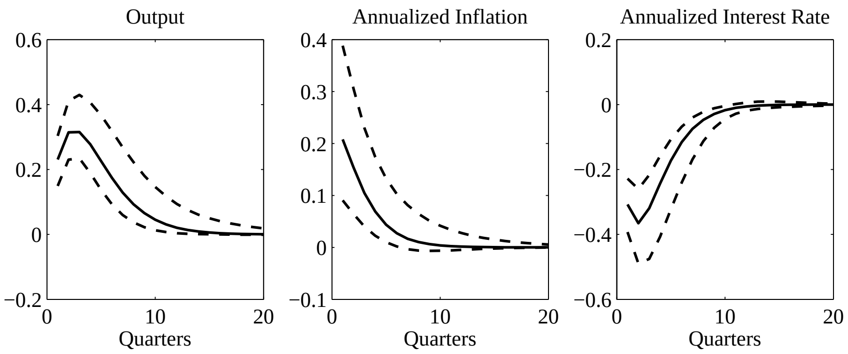
(a) Technology Shocks



(b) Preference Shocks



(c) Monetary Policy Shocks



Note: Plain line: median of the distribution, Dashed line: 95% Confidence interval

4 Conclusions

We have run a race involving three estimated versions of the NK model in order to to investigate the importance of alternative inertial mechanisms. While none of the three versions considered here is rejected by the data, there is a clear winner both in terms of overall fit and in terms of specific dynamic properties. The standard NK model without any real rigidities but with measurement errors in aggregate variables has the best performance. This finding is quite encouraging for the NK model, in the face of the widespread pessimism that has set in due to its increasing reliance on features such as adjustment costs and backward looking agents.

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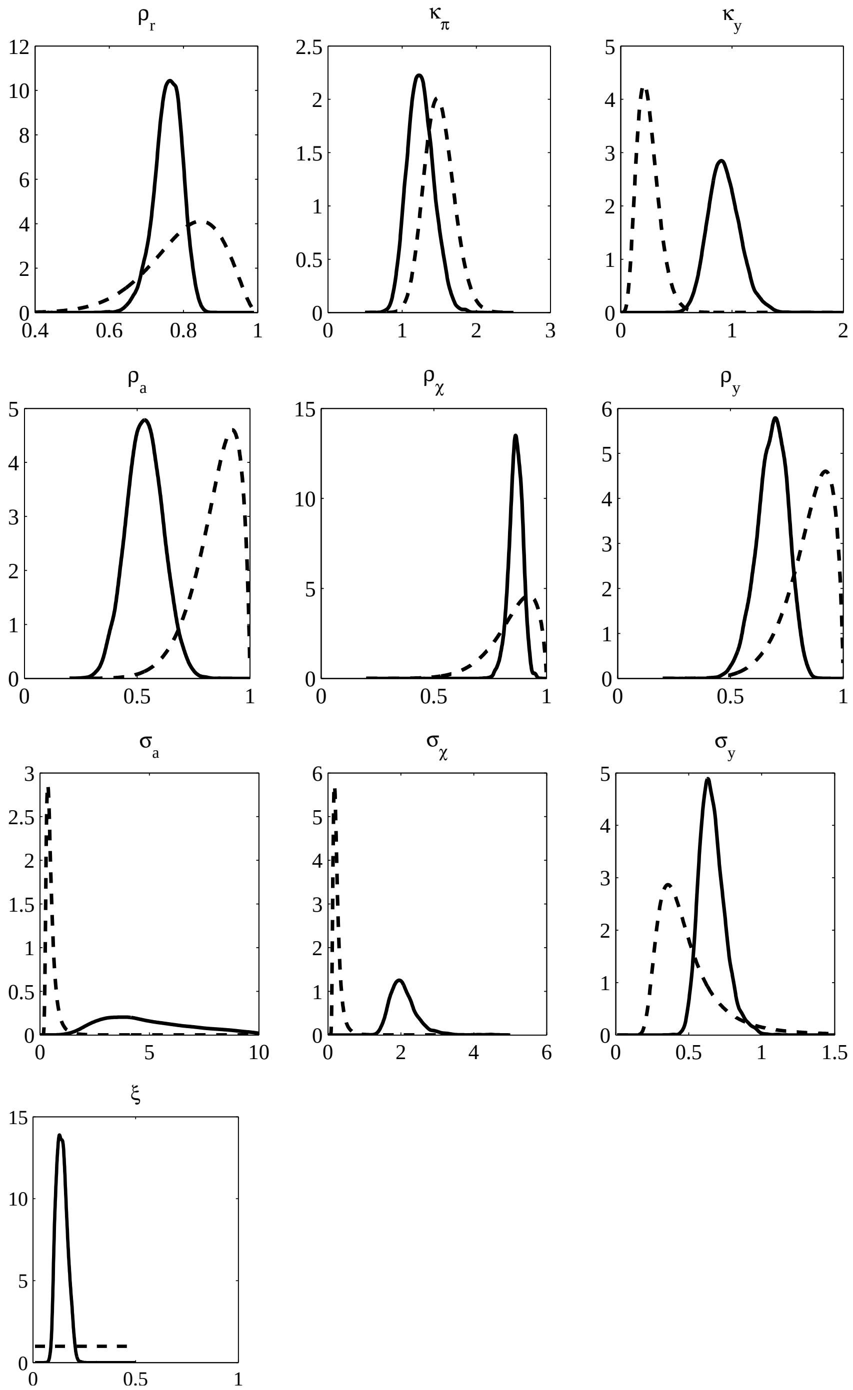
6 Appendices: (Not intended for publication)

6.1 Appendix 1

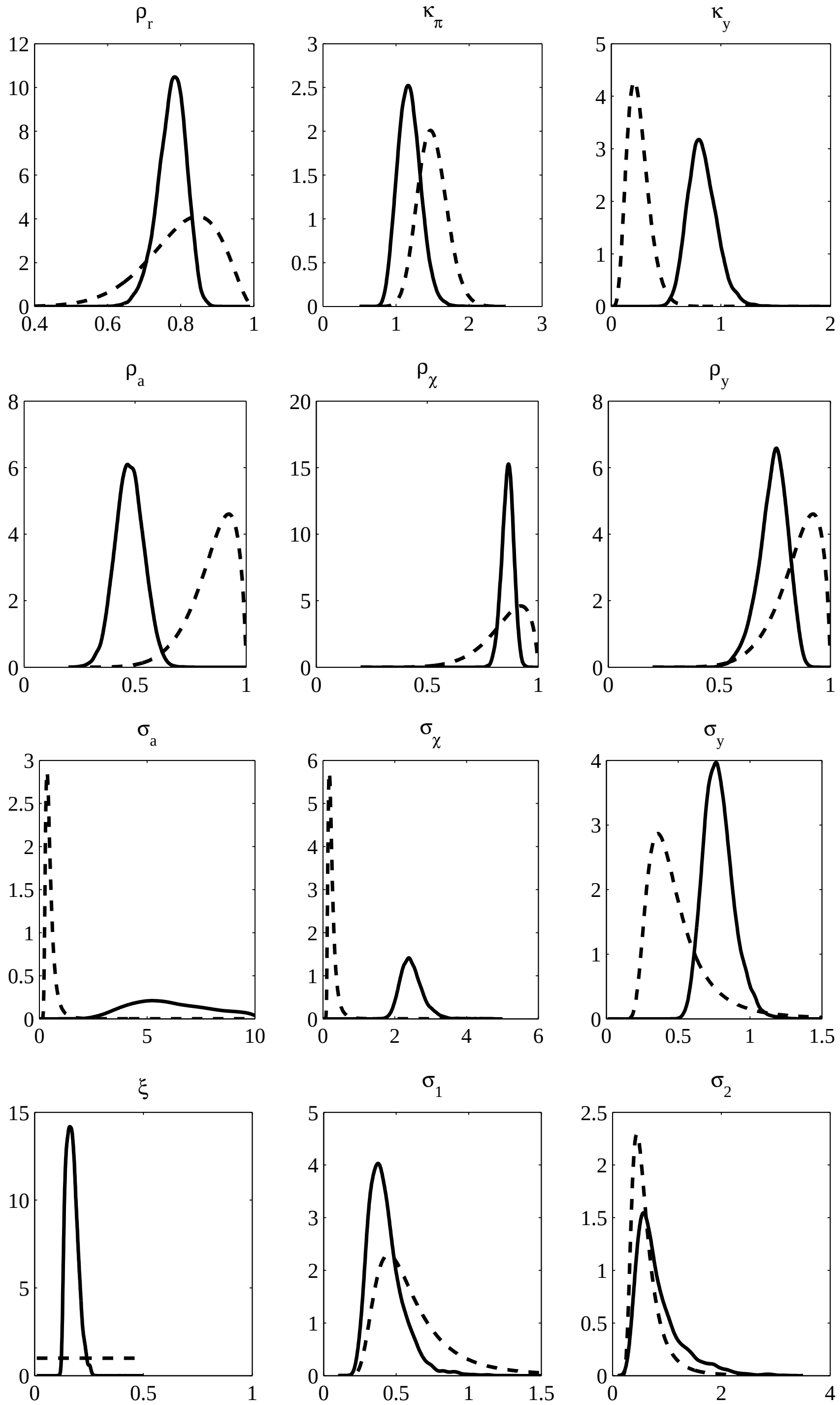
Prior Vs Posterior Densities

Figure 5: Frictionless Economy, Perfect Information



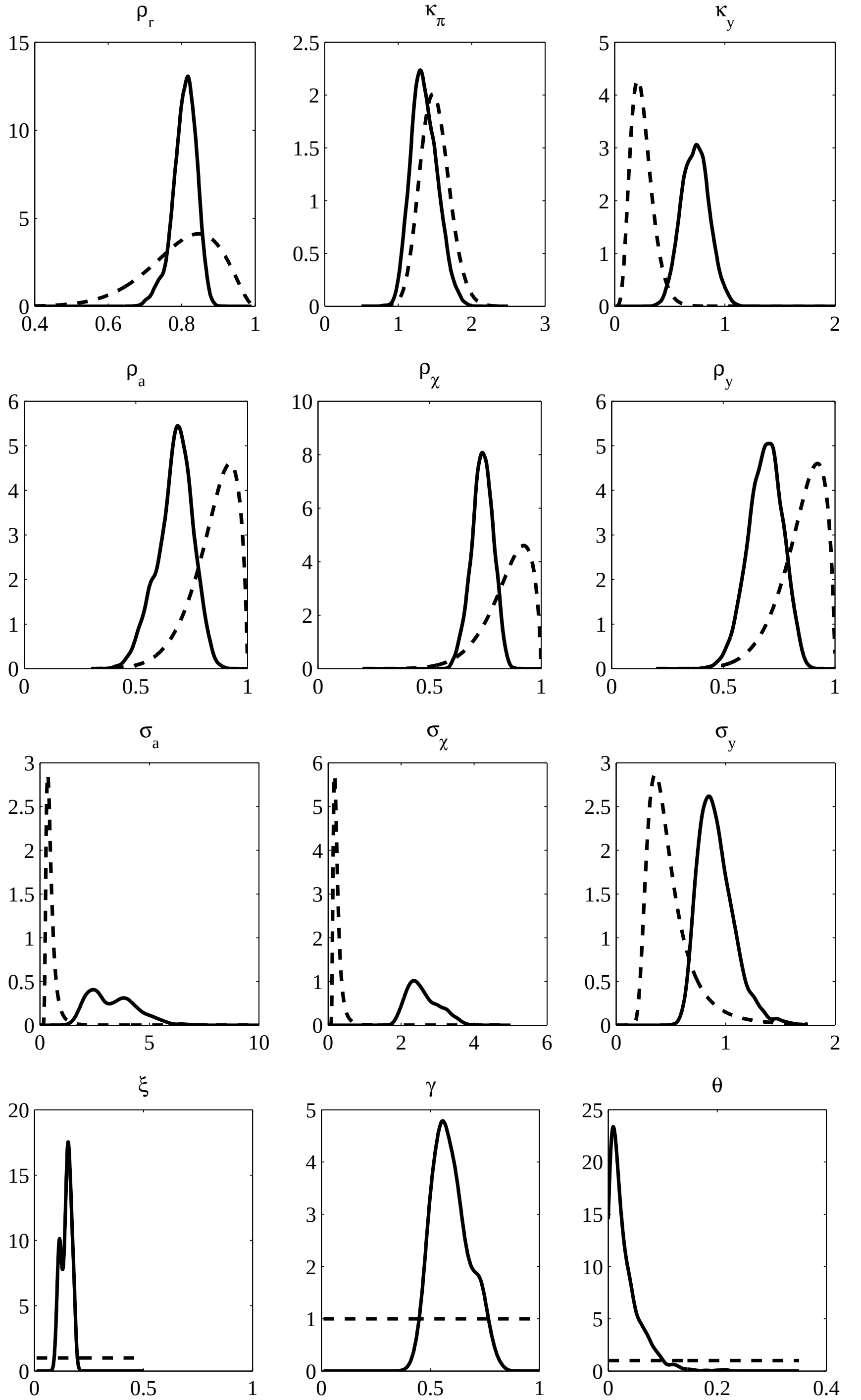
<u>Note:</u> Dashed line: Prior Distribution, Plain line: Posterior Distribution

Figure 6: Frictionless Economy, Imperfect Information



Note: Dashed line: Prior Distribution, Plain line: Posterior Distribution

Figure 7: Habit Persistence + Price Indexation, Perfect Information



Note: Dashed line: Prior Distribution, Plain line: Posterior Distribution

6.2 Appendix 2

6.2.1 The Model

Household: The household solves the following program

$$\max \mathbb{E}_t \sum_{\tau=0}^{\infty} \beta^{\tau} \chi_{t+\tau} \left[\log(c_{t+\tau} - \theta \overline{c}_{t+\tau-1}) + \psi_h(1 - h_{t+\tau})\right]$$

subject to

$$\int \varrho(t+1|t) q_{t+1} + B_t + P_t c_t = q_t + P_t w_t h_t + R_{t-1} B_{t-1} + \Pi_t$$

We have the following set of first order conditions

$$\chi_t (c_t - \theta \overline{c}_{t-1})^{-1} = \Lambda_t P_t \tag{16}$$

$$\chi_t \psi_h = \Lambda_t P_t w_t \tag{17}$$

$$\Lambda_t = \beta R_t \mathbb{E}_t \Lambda_{t+1} \tag{18}$$

$$\varrho(t+1|t)\Lambda_t = \beta \Lambda_{t+1} f_{t+1|t} \tag{19}$$

Final good: There exists a final good, y_t, which is produced by a representative firm by combining intermediate goods, $y_t(j)$, according to the following technology

$$y_t = \left(\int_0^1 y_t(i)^{\zeta} \mathrm{d}i\right)^{\frac{1}{\zeta}}$$

The optimal behavior of the final good firm yields the following demand function

$$y_t(i) = \left(\frac{P_t(i)}{P_t}\right)^{\frac{1}{\zeta-1}} y_t$$

and free entry on the market yields the aggregate price

$$P_t = \left(\int_0^1 P_t(i)^{\frac{\zeta}{\zeta-1}} \mathrm{d}i\right)^{\frac{\zeta-1}{\zeta}}$$

Intermediate good: The intermediate good is produced by means of labor according to the following constant returns to scale technology

$$y_t(i) = a_t h_t(i)$$

a_t is a macroeconomic technological shock.

Intermediate goods producers are monopolistically competitive, and therefore set prices for the good they produce. We follow **?** in assuming that firms set their prices for a stochastic number of periods. In each and every period, a firm either gets the chance to adjust its price (an event occurring with probability $(1-\xi)$) or it does not. If it does not get the chance, then it is assumed to set prices according to

$$P_t(i) = \pi_{t-1}^{\gamma}\overline{\pi}^{1-\gamma}P_{t-1}(i) \tag{20}$$

where $\gamma \in (0,1)$ determines the price indexation scheme.

On the other hand, a firm i that sets its price optimally in period t chooses a price, $P_t^\star$, in order to maximize:

$$\max_{P_t^\star(i)} \sum_{\tau=0}^{\infty} \int \varrho(t+\tau|t)(1-\gamma)^{\tau} \left(P_t^\star(i)\Xi_{t,\tau} - P_{t+\tau}s_{t+\tau}\right) y_{t+\tau}(i)$$

subject to the total demand and

$$\Xi_{t,\tau} = \begin{cases} \pi_t^{\gamma}\overline{\pi}^{1-\gamma} \times \ldots \times \pi_{t+\tau-1}^{\gamma}\overline{\pi}^{1-\gamma} & \text{if } \tau \geqslant 1 \\ 1 & \text{otherwise} \end{cases}$$

Note that we have $\Xi_{t,\tau+1} = \pi_t^{\gamma}\overline{\pi}^{1-\gamma}\Xi_{t+1,\tau}$.

Profit maximization can be rewritten as

$$\max_{P_t^\star} \sum_{\tau=0}^{\infty} \int \varrho(t+\tau|t)(1-\gamma)^{\tau} \left(P_t^\star\Xi_{t,\tau} - P_{t+\tau}s_{t+\tau}\right) \left(\frac{P_t^\star\Xi_{t,\tau}}{P_{t+\tau}}\right)^{\frac{1}{\zeta-1}} y_{t+\tau}$$

We therefore get immediately

$$P_t^\star = \frac{1}{\zeta}\frac{\sum_{\tau=0}^{\infty} \int \varrho(t+\tau|t)(1-\xi)^{\tau} P_{t+\tau}^{\frac{\zeta-2}{\zeta-1}} \Xi_{t,\tau}^{\frac{1}{\zeta-1}} s_{t+\tau}y_{t+\tau}}{\sum_{\tau=0}^{\infty} \int \varrho(t+\tau|t)(1-\xi)^{\tau} \Xi_{t,\tau}^{\frac{\zeta}{\zeta-1}} P_{t+\tau}^{\frac{1}{1-\zeta}} y_{t+\tau}}$$

which may be rewritten as

$$P_t^\star = \frac{1}{\zeta}\frac{N_t}{D_t}$$

where

$$N_t \equiv \sum_{\tau=0}^{\infty} \int \varrho(t+\tau|t)(1-\xi)^{\tau} P_{t+\tau}^{\frac{\zeta-2}{\zeta-1}} \Xi_{t,\tau}^{\frac{1}{\zeta-1}} s_{t+\tau}y_{t+\tau}$$
$$D_t \equiv \sum_{\tau=0}^{\infty} \int \varrho(t+\tau|t)(1-\xi)^{\tau} \Xi_{t,\tau}^{\frac{\zeta}{\zeta-1}} P_{t+\tau}^{\frac{1}{1-\zeta}} y_{t+\tau}$$

Using the fact that $\Xi_{t,\tau+1} = \pi_t^{\gamma}\overline{\pi}^{1-\gamma}\Xi_{t+1,\tau}$, the preceding system can be recursively stated as

$$N_t = P_t^{\frac{\zeta-2}{\zeta-1}} s_t y_t + (1-\xi)\left(\pi_t^{\gamma}\overline{\pi}^{1-\gamma}\right)^{\frac{1}{\zeta-1}} \int \varrho(t+1|t)N_{t+1}$$
$$D_t = P_t^{\frac{1}{1-\zeta}} y_t + (1-\xi)\left(\pi_t^{\gamma}\overline{\pi}^{1-\gamma}\right)^{\frac{\zeta}{\zeta-1}} \int \varrho(t+1|t)D_{t+1}$$

Since the price setting scheme is independent of any firm specific characteristic, all firms that reset their prices will choose the same price, such that $P_t^\star(i) = P_t^\star \ \forall i$.

In each period, a fraction ξ of contracts ends and $(1-\xi)$ survives. Hence, from the aggregate price index definition and the price mechanism, the aggregate price index writes

$$P_t = \left(\xi P_t^{\star\frac{\zeta}{\zeta-1}} + (1-\xi)(\pi_{t-1}^{\gamma}\overline{\pi}^{1-\gamma}P_{t-1})^{\frac{\zeta}{\zeta-1}} \right)^{\frac{\zeta-1}{\zeta}} \tag{21}$$

Equilibrium The labor market equilibrium is given by

$$\int_0^1 h_t(i)\mathrm{d}i = h_t$$

The good market equilibrium is

$$y_t = c_t$$

Let us define $\lambda_t = P_t\Lambda_t$, $\pi_t = P_t/P_{t-1}$, $\pi_t(j) = P_t(j)/P_{t-1}(j)$, $p_t(j) = P_t(j)/P_t$, $n_t \equiv N_t/P_t^{\frac{\zeta-2}{\zeta-1}}$ and $d_t \equiv D_t/P_t^{\frac{1}{1-\zeta}}$. The equilibrium writes

$$\begin{aligned}
y_t &= \left(\int_0^1 y_t(i)^\zeta \mathrm{d}i \right)^{\frac{1}{\zeta}} \\
y_t(i) &= a_t h_t(i) \\
h_t &= \int_0^1 h_t(i)\mathrm{d}i \\
\chi_t(y_t - \theta\overline{y}_{t-1})^{-1} &= \lambda_t \\
\chi_t\psi_h &= \lambda_t w_t \\
w_t &= s_t a_t \\
\lambda_t &= \beta R_t \mathbb{E}_t \frac{\lambda_{t+1}}{\pi_{t+1}} \\
\varrho(t+1|t)\lambda_t &= \beta \frac{\lambda_{t+1}}{\pi_{t+1}} f_{t+1|t} \\
n_t &= s_t y_t + \beta(1-\xi)\mathbb{E}_t \frac{\lambda_{t+1}}{\lambda_t} \left(\frac{\pi_t^\gamma \overline{\pi}^{1-\gamma}}{\pi_{t+1}} \right)^{\frac{1}{\zeta-1}} n_{t+1} \\
d_t &= y_t + \beta(1-\xi)\mathbb{E}_t \frac{\lambda_{t+1}}{\lambda_t} \left(\frac{\pi_t^\gamma \overline{\pi}^{1-\gamma}}{\pi_{t+1}} \right)^{\frac{\zeta}{\zeta-1}} d_{t+1} \\
1 &= \left(\int_0^1 p_t(i)^{\frac{\zeta}{\zeta-1}} \mathrm{d}i \right)^{\frac{\zeta-1}{\zeta}} = \left(\xi p_t^{\star\frac{\zeta}{\zeta-1}} + (1-\xi) \left(\frac{\pi_{t-1}^\gamma \overline{\pi}^{1-\gamma}}{\pi_t} \right)^{\frac{\zeta}{\zeta-1}} \right)^{\frac{\zeta-1}{\zeta}} \\
p_t^\star &= \frac{1}{\zeta} \frac{n_t}{d_t}
\end{aligned}$$

Log–linear Representation

$$\widehat{y}_t = \int_0^1 \widehat{y}_t(j)\mathrm{d}j \tag{22}$$

$$\widehat{y}_t(j) = \widehat{a}_t + \widehat{h}_t(j) \tag{23}$$

$$\widehat{h}_t = \int_0^1 \widehat{h}_t(j)\mathrm{d}j \tag{24}$$

$$\widehat{\chi}_t - \frac{1}{1-\theta}\widehat{y}_t + \frac{\theta}{1-\theta}\widehat{y}_{t-1} = \widehat{\lambda}_t \tag{25}$$

$$\widehat{\chi}_t = \widehat{\lambda}_t + \widehat{w}_t \tag{26}$$

$$\widehat{w}_t = \widehat{s}_t + \widehat{a}_t \tag{27}$$

$$\widehat{\lambda}_t = \mathbb{E}_t\widehat{\lambda}_{t+1} + \widehat{R}_t - \mathbb{E}_t\widehat{\pi}_{t+1} \tag{28}$$

$$\widehat{n}_t = (1-\beta(1-\xi))(\widehat{s}_t + \widehat{y}_t) + \beta(1-\xi)\mathbb{E}_t\left(\widehat{\lambda}_{t+1} - \widehat{\lambda}_t + \frac{\gamma}{\zeta-1}\widehat{\pi}_t - \frac{1}{\zeta-1}\widehat{\pi}_{t+1} + \widehat{n}_{t+1}\right) \tag{29}$$

$$\widehat{d}_t = (1-\beta(1-\xi))\widehat{y}_t + \beta(1-\xi)\mathbb{E}_t\left(\widehat{\lambda}_{t+1} - \widehat{\lambda}_t + \frac{\gamma\zeta}{\zeta-1}\widehat{\pi}_t - \frac{\zeta}{\zeta-1}\widehat{\pi}_{t+1} + \widehat{d}_{t+1}\right) \tag{30}$$

$$\xi\widehat{p}_t^\star + (1-\xi)(\gamma\widehat{\pi}_{t-1} - \widehat{\pi}_t) = 0 \tag{31}$$

$$\widehat{p}_t^\star = \widehat{n}_t - \widehat{d}_t \tag{32}$$

6.2.2 Perfect Information Case

Combining (29)–(30) and using (32), we obtain

$$\widehat{p}_t^\star = (1-\beta(1-\xi))\widehat{s}_t + \beta(1-\xi)\mathbb{E}_t\left(\widehat{p}_t^\star + \widehat{\pi}_{t+1} - \gamma\widehat{\pi}_t\right) \tag{33}$$

Finally, combining (33) and (31), we obtain the log–linear version of inflation dynamics

$$\widehat{\pi}_t = \frac{\xi(1-\beta(1-\xi))}{(1-\xi)(1+\beta\gamma)}\widehat{s}_t + \frac{\gamma}{1+\beta\gamma}\widehat{\pi}_{t-1} + \frac{\beta}{1+\beta\gamma}\mathbb{E}_t\widehat{\pi}_{t+1} \tag{34}$$

Using the good market equilibrium together with (25) and (28), we obtain the IS curve

$$\widehat{y}_t = \frac{\theta}{1+\theta}\widehat{y}_{t-1} + \frac{1}{1+\theta}\mathbb{E}_t\widehat{y}_{t+1} - \frac{1-\theta}{(1+\theta)}(\widehat{R}_t - \mathbb{E}_t\widehat{\pi}_{t+1}) + \frac{1-\theta}{(1+\theta)}(\widehat{\chi}_t - \mathbb{E}_t\widehat{\chi}_{t+1}) \tag{35}$$

Combining (26), (22), (23) and (27), we obtain

$$\widehat{s}_t = \left(\frac{1}{1-\theta}\right)\widehat{y}_t - \frac{\theta}{1-\theta}\widehat{y}_{t-1} - \widehat{a}_t \tag{36}$$

Therefore, the new Keynesian Phillips curve writes

$$\begin{aligned}\widehat{\pi}_t =& \frac{\gamma}{1+\beta\gamma}\widehat{\pi}_{t-1} + \frac{\beta}{1+\beta\gamma}\mathbb{E}_t\widehat{\pi}_{t+1} + \frac{\xi(1-\beta(1-\xi))}{(1-\xi)(1+\beta\gamma)}\left(\frac{1}{1-\theta}\right)\widehat{y}_t \\ &- \frac{\xi(1-\beta(1-\xi))}{(1-\xi)(1+\beta\gamma)}\frac{\theta}{1-\theta}\widehat{y}_{t-1} - \frac{\xi(1-\beta(1-\xi))}{(1-\xi)(1+\beta\gamma)}\widehat{a}_t\end{aligned} \tag{37}$$

The Taylor rule completes the description of the model.

6.2.3 The model with signal extraction

There are –at least– two alternative specifications of the imperfect information (signal extraction) problem. One specification involves the distinction between idiosyncratic and aggregate shocks. Suppose that the agents are subject to shocks that contain both idiosyncratic and common –aggregate– components and that the agents can only observe the combined shocks. If these two components have different stochastic processes then the agents need to solve a signal extraction problem .

An alternative and simpler specification involves the assumption that all shocks are common but they are measured with error. Of course, this statement is technically equivalent to assuming that a suitable subset of the endogenous variables is measured with error. Otherwise, knowledge of the model would allow the agents to solve out for the true values of the shocks, eliminating the signal extraction problem. This is the standard practice in the literature (for instance, see Svensson and Woodford, 2003). Some may find the assumption that the individuals may lack perfect knowledge of some of their own variables questionable. But it can be defended on the basis that, for instance, even at the firm level the output and/or the inputs may not be measured contemporaneously without any error. This is precisely the assumption made in models of sticky information or inattentive agents. We have opted for this specification because of two reasons: First, its empirical implementation is straightforward as it only requires the specification of the signals and the noise in the measurement of the variables. And second, given the existence of real time data (for instance, at the Philadelphia FED) one can assess the plausibility of the estimated amount of noise in the model by comparing it to that present, say, in data revisions. A specification with idiosyncratic shocks, on the other hand, may require knowledge about (or assumptions on) the relative variance of idiosyncratic and aggregate shocks in the estimation of the model.

In what follows, we assume that the productivity and preference shocks cannot be observed directly and can only be inferred from noisy signals that are available on output (or the output gap) and inflation. The agents make decisions and form expectations based on this information set. We denote by $\mathscr{E}_t$ the expectation operator in this case. The log–linear representation of the equilibrium is given by

$$\widetilde{y}_t = \int_0^1 \widetilde{y}_t(j)\mathrm{d}j \tag{38}$$

$$\widetilde{y}_t(j) = \widetilde{a}_t + \widetilde{h}_t(j) \tag{39}$$

$$\widetilde{h}_t = \int_0^1 \widetilde{h}_t(j)\mathrm{d}j \tag{40}$$

$$\widetilde{\chi}_t - \frac{1}{1-\theta}\widetilde{y}_t + \frac{\theta}{1-\theta}\widetilde{y}_{t-1} = \widetilde{\lambda}_t \tag{41}$$

$$\widetilde{\chi}_t = \widetilde{\lambda}_t + \widetilde{w}_t \tag{42}$$

$$\widetilde{w}_t = \widetilde{s}_t + \widetilde{a}_t \tag{43}$$

$$\widetilde{\lambda}_t = \mathscr{E}_t\widetilde{\lambda}_{t+1} + \widetilde{R}_t - \mathscr{E}_t\widetilde{\pi}_{t+1} \tag{44}$$

$$\widetilde{n}_t = (1-\beta(1-\xi))(\widetilde{s}_t + \widetilde{y}_t) + \beta(1-\xi)\mathscr{E}_t\left(\widetilde{\lambda}_{t+1} - \widetilde{\lambda}_t + \frac{\gamma}{\zeta-1}\widetilde{\pi}_t - \frac{1}{\zeta-1}\widetilde{\pi}_{t+1} + \widetilde{n}_{t+1}\right) \tag{45}$$

$$\widetilde{d}_t = (1-\beta(1-\xi))\widetilde{y}_t + \beta(1-\xi)\mathscr{E}_t\left(\widetilde{\lambda}_{t+1} - \widetilde{\lambda}_t + \frac{\gamma\zeta}{\zeta-1}\widetilde{\pi}_t - \frac{\zeta}{\zeta-1}\widetilde{\pi}_{t+1} + \widetilde{d}_{t+1}\right) \tag{46}$$

$$\xi\widetilde{p}_t^\star + (1-\xi)(\gamma\widetilde{\pi}_{t-1} - \widetilde{\pi}_t) = 0 \tag{47}$$

$$\widetilde{p}_t^\star = \widetilde{n}_t - \widetilde{d}_t \tag{48}$$

$\widetilde{x}_t \equiv \widehat{x}_t + \xi_t^x$ denotes observed variables, where $\widehat{x}_t$ denotes the true value of x_t and ξ_t^x is an associated measurement error. Then, the system may be rewritten in the simpler form

$$\widehat{y}_t = \frac{\theta}{1+\theta}\widehat{y}_{t-1} + \frac{1}{1+\theta}\mathscr{E}_t\widehat{y}_{t+1} - \frac{1-\theta}{(1+\theta)}(\widehat{R}_t - \mathscr{E}_t\widehat{\pi}_{t+1}) + \frac{1-\theta}{(1+\theta)}(\widehat{\chi}_t - \mathscr{E}_t\widehat{\chi}_{t+1}) \tag{49}$$

$$\begin{aligned}\widehat{\pi}_t = {} & \frac{\gamma}{1+\beta\gamma}\widehat{\pi}_{t-1} + \frac{\beta}{1+\beta\gamma}\mathscr{E}_t\widehat{\pi}_{t+1} + \frac{\xi(1-\beta(1-\xi))}{(1-\xi)(1+\beta\gamma)}\left(\frac{1}{1-\theta}\right)\widehat{y}_t \\ & - \frac{\xi(1-\beta(1-\xi))}{(1-\xi)(1+\beta\gamma)}\frac{\theta}{1-\theta}\widehat{y}_{t-1} - \frac{\xi(1-\beta(1-\xi))}{(1-\xi)(1+\beta\gamma)}\widehat{a}_t\end{aligned} \tag{50}$$

$$\widetilde{R}_t = \rho_r\widetilde{R}_{t-1} + (1-\rho_r)(\kappa_y(\widetilde{y}_t) - \widetilde{y}_t)) + \kappa_\pi\widetilde{\pi}_t + \widetilde{\nu}_t \tag{51}$$

Let the state of the economy be represented by two vectors $\widetilde{X}_t^b$ and $\widetilde{X}_t^f$. The first one includes the predetermined (backward looking) state variables, i.e. $\widetilde{X}_t^b = (\widetilde{R}_{t-1}, \widetilde{z}_t, \widetilde{g}_t, \widetilde{\varepsilon}_t^R)'$, whereas the second one consists of the forward looking state variables, i.e. $\widetilde{X}_t^f = (\widetilde{y}_t, \widetilde{\pi}_t)'$. The model admits the following representation

$$M_0 \begin{pmatrix} \widetilde{X}_{t+1}^b \\ \mathbb{E}_t \widetilde{X}_{t+1}^f \end{pmatrix} + M_1 \begin{pmatrix} \widetilde{X}_t^b \\ \widetilde{X}_t^f \end{pmatrix} = M_2 \varepsilon_{t+1} \tag{52}$$

where

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -\tau & 0 & 0 & 0 & 1 & \tau \\ 0 & 0 & 0 & 0 & 0 & \beta \end{pmatrix}$$

$$M_1 = \begin{pmatrix} -\rho_R & 0 & 0 & -1 & -(1-\rho_R)\psi_y & -(1-\rho_R)\psi_\pi \\ 0 & -\rho_z & 0 & 0 & 0 & 0 \\ 0 & 0 & -\rho_g & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & -\kappa & 0 & 0 & \kappa & -1 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\varepsilon_{t+1} = \{\varepsilon_{t+1}^z, \varepsilon_{t+1}^g, \varepsilon_{t+1}^R\}$$

Thus the first row corresponds to the Taylor rule, the second, third and forth row to the demand, cost push shock and policy shock, the fifth row to the IS-curve and the sixth row to the Phillips curve. Let us denote the signal process by $\{S_t\}$. The measurement equation relates the state of the economy to the signal:

$$S_t = C \begin{pmatrix} \widetilde{X}_t^b \\ \widetilde{X}_t^f \end{pmatrix} + v_t. \tag{53}$$

Finally u and v are assumed to be normally distributed covariance matrices Σ_{uu} and Σ_{vv} respectively and $E(uv') = 0$.

$X_{t+i|t} = E(X_{t+i}|\mathscr{I}_t)$ for $i \geqslant 0$ and where $\mathscr{I}_t$ denotes the information set available to the agents at the beginning of period t. The information set available to the agents consists of *i)* the structure of the model and *ii)* the history of the observable signals they are given in each period:

$$\mathscr{I}_t = \{S_{t-j}, j \geqslant 0, M_0, M_1, M_2, C, \Sigma_{uu}, \Sigma_{vv}\}$$

The information structure of the agents is described fully by the specification of the signals.

6.2.4 Solving the system

Step 1: We first solve for the expected system:

$$M_0 \begin{pmatrix} X^b_{t+1|t} \\ X^f_{t+1|t} \end{pmatrix} + M_1) \begin{pmatrix} X^b_{t|t} \\ X^f_{t|t} \end{pmatrix} = \tag{54}$$

which rewrites as

$$\begin{pmatrix} X^b_{t+1|t} \\ X^f_{t+1|t} \end{pmatrix} = W \begin{pmatrix} X^b_{t|t} \\ X^f_{t|t} \end{pmatrix} \tag{55}$$

where

$$W = -M_0^{-1} M_1$$

After getting the Jordan form associated to (55) and applying standard methods for eliminating bubbles, we get

$$X^f_{t|t} = G X^b_{t|t}$$

From which we get

$$X^b_{t+1|t} = (W_{bb} + W_{bf} G) X^b_{t|t} = W^b X^b_{t|t} \tag{56}$$

$$X^f_{t+1|t} = (W_{fb} + W_{ff} G) X^b_{t|t} = W^f X^b_{t|t} \tag{57}$$

Step 2: We have

$$M_0 \begin{pmatrix} X^b_{t+1} \\ X^f_{t+1|t} \end{pmatrix} + M_1 \begin{pmatrix} X^b_{t} \\ X^f_{t} \end{pmatrix} = M_2 u_{t+1}$$

Taking expectations, we have

$$M_0 \begin{pmatrix} X^b_{t+1|t} \\ X^f_{t+1|t} \end{pmatrix} + M_1 \begin{pmatrix} X^b_{t|t} \\ X^f_{t|t} \end{pmatrix} = 0$$

Subtracting, we get

$$M_0 \begin{pmatrix} X^b_{t+1} - X^b_{t+1|t} \\ 0 \end{pmatrix} + M_1 \begin{pmatrix} X^b_{t} - X^b_{t|t} \\ X^f_{t} - X^f_{t|t} \end{pmatrix} = M_2 u_{t+1} \tag{58}$$

which rewrites

$$\begin{pmatrix} X^b_{t+1} - X^b_{t+1|t} \\ 0 \end{pmatrix} = W^c \begin{pmatrix} X^b_{t} - X^b_{t|t} \\ X^f_{t} - X^f_{t|t} \end{pmatrix} + M_0^{-1} M_2 u_{t+1} \tag{59}$$

where, $W^c = -M_0^{-1} M_1$. Hence, considering the second block of the above matrix equation, we get

$$W^c_{fb}(X^b_t - X^b_{t|t}) + W^c_{ff}(X^f_t - X^f_{t|t}) = 0$$

which gives

$$X^f_t = F^0 X^b_t + F^1 X^b_{t|t}$$

with $F^0 = -{W^c_{ff}}^{-1} W^c_{fb}$ and $F^1 = G - F^0$.

Now considering the first block we have

$$X^b_{t+1} = X^b_{t+1|t} + W^c_{bb}(X^b_t - X^b_{t|t}) + W^c_{bf}(X^f_t - X^f_{t|t}) + M^2 u_{t+1}$$

from which we get, using (56)

$$X^b_{t+1} = M^0 X^b_t + M^1 X^b_{t|t} + M^2 u_{t+1}$$

with $M^0 = W^c_{bb} + W^c_{bf} F^0$, $M^1 = W^b - M^0$ and $M^2 = M_0^{-1} M_2$.

We also have

$$S_t = C_b X^b_t + C_f X^f_t + v_t$$

from which we get

$$S_t = S^0 X^b_t + S^1 X^b_{t|t} + v_t$$

where $S^0 = C_b + C_f F^0$ and $S^1 = C_f F^1$

6.2.5 Filtering

Since our solution involves terms in $X^b_{t|t}$, we would like to compute this quantity. However, the only information we can exploit is a signal S_t that we described previously. We therefore use a Kalman filter approach to compute the optimal prediction of $X^b_{t|t}$.

In order to recover the Kalman filter, it is a good idea to think in terms of expectation errors. Therefore, let us define

$$\widetilde{X}^b_t = X^b_t - X^b_{t|t-1}$$

and

$$\widetilde{S}_t = S_t - S_{t|t-1}$$

Note that since S_t depends on $X^b_{t|t}$, only the signal relying on $\widetilde{S}_t = S_t - S^1 X^b_{t|t}$ can be used to infer anything on $X^b_{t|t}$. Therefore, the policy maker revises its expectations using a linear rule depending on $\widetilde{S}^e_t = S_t - S^1 X^b_{t|t}$. The filtering equation then writes

$$X^b_{t|t} = X^b_{t|t-1} + K(\widetilde{S}^e_t - \widetilde{S}^e_{t|t-1}) = X^b_{t|t-1} + K(S^0 \widetilde{X}^b_t + v_t)$$

where K is the filter gain matrix, that we would like to compute.

The first thing we have to do is to rewrite the system in terms of state–space representation. Since $S_{t|t-1} = (S^0 + S^1)X^b_{t|t-1}$, we have

$$\begin{aligned}\widetilde{S}_t &= S^0(X^b_t - X^b_{t|t}) + S^1(X^b_{t|t} - X^b_{t|t-1}) + v_t \\ &= S^0\widetilde{X}^b_t + S^1K(S^0\widetilde{X}^b_t + v_t) + v_t \\ &= S^\star\widetilde{X}^b_t + \nu_t\end{aligned}$$

where $S^\star = (I + S^1K)S^0$ and $\nu_t = (I + S^1K)v_t$.

Now, consider the law of motion of backward state variables, we get

$$\begin{aligned}\widetilde{X}^b_{t+1} &= M^0(X^b_t - X^b_{t|t}) + M^2u_{t+1} \\ &= M^0(X^b_t - X^b_{t|t-1} - X^b_{t|t} + X^b_{t|t-1}) + M^2u_{t+1} \\ &= M^0\widetilde{X}^b_t - M^0(X^b_{t|t} + X^b_{t|t-1}) + M^2u_{t+1} \\ &= M^0\widetilde{X}^b_t - M^0K(S^0\widetilde{X}^b_t + v_t) + M^2u_{t+1} \\ &= M^\star\widetilde{X}^b_t + \omega_{t+1}\end{aligned}$$

where $M^\star = M^0(I - KS^0)$ and $\omega_{t+1} = M^2u_{t+1} - M^0Kv_t$.

We therefore end–up with the following state–space representation

$$\widetilde{X}^b_{t+1} = M^\star\widetilde{X}^b_t + \omega_{t+1} \quad (60)$$

$$\widetilde{S}_t = S^\star\widetilde{X}^b_t + \nu_t \quad (61)$$

For which the Kalman filter is given by

$$\widetilde{X}^b_{t|t} = \widetilde{X}^b_{t|t-1} + PS^{\star\prime}(S^\star PS^{\star\prime} + \Sigma_{\nu\nu})^{-1}(S^\star\widetilde{X}^b_t + \nu_t)$$

But since $\widetilde{X}^b_{t|t}$ is an expectation error, it is not correlated with the information set in $t-1$, such that $\widetilde{X}^b_{t|t-1} = 0$. The prediction formula for $\widetilde{X}^b_{t|t}$ therefore reduces to

$$\widetilde{X}^b_{t|t} = PS^{\star\prime}(S^\star PS^{\star\prime} + \Sigma_{\nu\nu})^{-1}(S^\star\widetilde{X}^b_t + \nu_t) \quad (62)$$

where P solves

$$P = M^\star PM^{\star\prime} + \Sigma_{\omega\omega}$$

and $\Sigma_{\nu\nu} = (I + S^1K)\Sigma_{vv}(I + S^1K)'$ and $\Sigma_{\omega\omega} = M^0K\Sigma_{vv}K'M^{0\prime} + M^2\Sigma_{uu}M^{2\prime}$

Note however that the above solution is obtained for a given K matrix that remains to be computed. We can do that by using the basic equation of the Kalman filter:

$$\begin{aligned}X^b_{t|t} &= X^b_{t|t-1} + K(\widetilde{S}^e_t - \widetilde{S}^e_{t|t-1}) \\ &= X^b_{t|t-1} + K(S_t - S^1X^b_{t|t} - (S_{t|t-1} - S^1X^b_{t|t-1})) \\ &= X^b_{t|t-1} + K(S_t - S^1X^b_{t|t} - S^0X^b_{t|t-1})\end{aligned}$$

Solving for $X^b_{t|t}$, we get

$$\begin{aligned}
X^b_{t|t} &= (I+KS^1)^{-1}(X^b_{t|t-1}+K(S_t-S^0X^b_{t|t-1})) \\
&= (I+KS^1)^{-1}(X^b_{t|t-1}+KS^1X^b_{t|t-1}-KS^1X^b_{t|t-1}+K(S_t-S^0X^b_{t|t-1})) \\
&= (I+KS^1)^{-1}(I+KS^1)X^b_{t|t-1}+(I+KS^1)^{-1}K(S_t-(S^0+S^1)X^b_{t|t-1})) \\
&= X^b_{t|t-1}+(I+KS^1)^{-1}K\widetilde{S}_t \\
&= X^b_{t|t-1}+K(I+S^1K)^{-1}\widetilde{S}_t \\
&= X^b_{t|t-1}+K(I+S^1K)^{-1}(S^\star\widetilde{X}^b_t+\nu_t)
\end{aligned}$$

where we made use of the identity $(I+KS^1)^{-1}K \equiv K(I+S^1K)^{-1}$. Hence, identifying to (62), we have

$$K(I+S^1K)^{-1} = PS^{\star\prime}(S^\star PS^{\star\prime}+\Sigma_{\nu\nu})^{-1}$$

remembering that $S^\star=(I+S^1K)S^0$ and $\Sigma_{\nu\nu}=(I+S^1K)\Sigma_{vv}(I+S^1K)'$, we have

$$K(I+S^1K)^{-1} = PS^{0'}(I+S^1K)'((I+S^1K)S^0PS^{0'}(I+S^1K)'+(I+S^1K)\Sigma_{vv}(I+S^1K)')^{-1}(I+S^1K)S^0$$

which rewrites as

$$\begin{aligned}
K(I+S^1K)^{-1} &= PS^{0'}(I+S^1K)'\left[(I+S^1K)(S^0PS^{0'}+\Sigma_{vv})(I+S^1K)'\right]^{-1} \\
K(I+S^1K)^{-1} &= PS^{0'}(I+S^1K)'(I+S^1K)'^{-1}(S^0PS^{0'}+\Sigma_{vv})^{-1}(I+S^1K)^{-1}
\end{aligned}$$

Hence, we obtain

$$K = PS^{0'}(S^0PS^{0'}+\Sigma_{vv})^{-1} \tag{63}$$

Now, recall that

$$P = M^\star PM^{\star\prime}+\Sigma_{\omega\omega}$$

Remembering that $M^\star = M^0(I+KS^0)$ and $\Sigma_{\omega\omega} = M^0K\Sigma_{vv}K'M^{0'}+M^2\Sigma_{uu}M^{2'}$, we have

$$\begin{aligned}
P &= M^0(I-KS^0)P\left[M^0(I-KS^0)\right]'+M^0K\Sigma_{vv}K'M^{0'}+M^2\Sigma_{uu}M^{2'} \\
&= M^0\left[(I-KS^0)P(I-S^{0'}K')+K\Sigma_{vv}K'\right]M^{0'}+M^2\Sigma_{uu}M^{2'}
\end{aligned}$$

Plugging the definition of K in the latter equation, we obtain

$$P = M^0\left[P-PS^{0'}(S^0PS^{0'}+\Sigma_{vv})^{-1}S^0P\right]M^{0'}+M^2\Sigma_{uu}M^{2'} \tag{64}$$

7 Summary

We finally end–up with the system of equations:

$$X^b_{t+1} = M^0 X^b_t + M^1 X^b_{t|t} + M^2 u_{t+1} \tag{65}$$

$$S_t = S^0_b X^b_t + S^1_b X^b_{t|t} + v_t \tag{66}$$

$$X^f_t = F^0 X^b_t + F^1 X^b_{t|t} \tag{67}$$

$$X^b_{t|t} = X^b_{t|t-1} + K(S^0(X^b_t - X^b_{t|t-1}) + v_t) \tag{68}$$

$$X^b_{t+1|t} = (M^0 + M^1) X^b_{t|t} \tag{69}$$

to describe the dynamics of our economy.

This may be recast as a standard state–space problem

$$\begin{aligned} X^b_{t+1|t+1} &= X^b_{t+1|t} + K(S^0(X^b_{t+1} - X^b_{t+1|t}) + v_{t+1}) \\ &= (M^0 + M^1) X^b_{t|t} + K(S^0(M^0 X^b_t + M^1 X^b_{t|t} + M^2 u_{t+1} - (M^0 + M^1) X^b_{t|t}) + v_{t+1}) \\ &= K S^0 M^0 X^b_t + ((I - K S^0) M^0 + M^1) X^b_{t|t} + K S^0 M^2 u_{t+1} + K v_{t+1} \end{aligned}$$

Then

$$\begin{pmatrix} X^b_{t+1} \\ X^b_{t+1|t+1} \end{pmatrix} = M_{\text{X}} \begin{pmatrix} X^b_t \\ X^b_{t|t} \end{pmatrix} + M_{\text{E}} \begin{pmatrix} u_{t+1} \\ v_{t+1} \end{pmatrix}$$

where

$$M_{\text{X}} = \begin{pmatrix} M^0 & M^1 \\ K S^0 M^0 & ((I - K S^0) M^0 + M^1) \end{pmatrix} \text{ and } M_{\text{E}} = \begin{pmatrix} M^2 & 0 \\ K S^0 M^2 & K \end{pmatrix}$$

and

$$X^f_t = M_{\text{F}} \begin{pmatrix} X^b_t \\ X^b_{t|t} \end{pmatrix}$$

where

$$M_{\text{F}} = \begin{pmatrix} F^0 & F^1 \end{pmatrix}$$