1 Advanced Methods in Macroeconomics

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1.1 Solving dynamic models

1.1.1 A Model with an analytical Solution

Let the utility function of the representative agent be logarithmic and the production function Cobb-Douglas

$$\begin{aligned} u_t &= \log(C_t) \\ Y_t &= Z_t K_t^{\alpha} \end{aligned} \tag{1}$$

where Z_t is an iid shock with mean equal to unity. Let the depreciation rate of physical capital be equal to one: $\delta = 1$.

The representative agent maximizes

$$\max\ E_t \left\{ \sum_{t=0}^{\infty} \beta^t u_t(C_t) \right\} \tag{2}$$

$$s.t.\quad \begin{aligned} Y_t &= C_t + I_t \\ K_{t+1} &= I_t \end{aligned} \tag{3}$$

The maximization problem can be re-written as

$$\max\ V(K_t) = \max \{\log(C_t) + \beta E_t [V(K_{t+1})]\} \tag{4}$$

$$s.t.\ Z_t K_t^{\alpha} = C_t + I_t = C_t + K_{t+1} \tag{5}$$

The First-Order-Conditions (FOC) are:

$$C_t : \ \frac{1}{C_t} = \lambda_t \tag{6}$$

$$K_{t+1} : \ \beta E \left\{ \frac{dV}{dK_{t+1}} \right\} = \lambda_t \tag{7}$$

where λ is the Langrange multiplier. The derivative with regard to K_{t+1} is calculated by taking the derivative of the budget constraint with regard to K_t -inclusive of the Langrange multiplier- and updating to $t+1$. This yields

$$\beta E_t \left[\alpha Z_{t+1} K_{t+1}^{\alpha-1} \lambda_{t+1} \right] = \lambda_t \tag{8}$$

Subsequently,

$$\beta E_t \left[\frac{\alpha Z_{t+1} K_{t+1}^{\alpha-1}}{C_{t+1}} \right] = \frac{1}{C_t} \tag{9}$$

The purpose is to express the control variables $C_t; Y_t; K_{t+1}$ as functions of the state variables $A_t; K_t$. Given the functional forms assumed for utility and production and the fact all capital depreciates within one period, there exists a very simple solution for the control variables. To derive the solution, we "guess" that consumption (and also investment) is a constant share of income in each period and then verify the guess. That is, we assume

$$C_t = \eta Z_t K_t^{\alpha} \text{ for all } t = 0; 1; 2::: \tag{10}$$

The objective is to determine the unknown constant, η: Plugging (1.10) into (1.9) yields

$$\beta E_t \left[\frac{\alpha}{\eta K_{t+1}} \right] = \frac{1}{\eta Z_t K_t^{\alpha}} \tag{11}$$

In (11) we can drop the expectational term since all variables involved are known at t. We can then express the capital stock K_{t+1} as

$$K_{t+1} = \alpha\beta Z_t K_t^{\alpha} \tag{12}$$

Remember that K_{t+1} is equal to I_t: Inserting (10) and (12) into the budget constraint (5) gives

$$Z_t K_t^{\alpha} = \eta Z_t K_t^{\alpha} + \alpha\beta Z_t K_t^{\alpha}$$

which yields

$$\eta = 1 - \alpha\beta \tag{13}$$

Taking logarithms in the production function yields

$$\log(Y_t) = \log(Z_t) + \alpha \log(K_t) \tag{14}$$

We substitute (12) for K_t

$$\log(Y_t) = \log(Z_t) + \alpha \log(\alpha\beta Z_{t-1} K_{t-1}^{\alpha}) = \tag{15}$$

$$\log(Y_t) = \xi + \log(Z_t) + \alpha \log Y_{t-1} \text{ where } \xi = \alpha \log \alpha\beta \tag{16}$$

In the next step we describe the dynamics of the system around the -deterministic- steady state. In the steady state we have

$$Y_t = Y \quad Z_t = Z \quad \forall t \tag{17}$$

hence $\log Y = \xi/(1-a)$:

$$\xi = (1-\alpha)\log Y \tag{18}$$

Now we substitute (18) for ξ in (15) and rearrange to get the formula

$$\log\left(\frac{Y_t}{Y}\right) = \log\left(\frac{Z_t}{Z}\right) + \alpha\log\left(\frac{Y_{t-1}}{Y}\right) \tag{19}$$

which defines the percentage deviation of income from the steady state as function of past income and current technology deviations. Let a ^ denote percentage deviations from the steady state. We can write (19) as

$$\hat{y}_t = \hat{z}_t + \alpha\hat{y}_{t-1} \tag{20}$$

(1.16) provides an econometrically testable equation.

1.1.2 A Model with an Approximate Analytical Solution

If $\delta \neq 1$ then there exists no analytical solution. In such a case it may still be possible to derive an approximate solution. We will show how this can be done using the model of the previous section (for simplicity we will assume that $\delta = 0$).

Let the production function be $Y_t = (L_t Z_t)^{1-\alpha} K_t^{\alpha}$. Suppose $L_t = 1$. It follows that

$$Y_t = Z_t^{1-\alpha} K_t^{\alpha} \tag{21}$$

The value function is then given by

$$V(K_t; Z_t) = \max\{\log(C_t) + \beta E_t[V(K_{t+1}; Z_{t+1})]\} \text{ subject to} \tag{22}$$

$$Z_t^{1-\alpha} K_t^{\alpha} - C_t - (K_{t+1} - K_t) = 0 \tag{23}$$

The FOC that must hold in equilibrium are

$$\frac{1}{C_t} - \lambda_t = 0 \tag{24}$$

$$\beta E_t\left[\lambda_{t+1}(\alpha Z_{t+1}^{1-\alpha} K_{t+1}^{\alpha-1} + 1)\right] = \lambda_t \tag{25}$$

Combining (24) and (25) gives

$$\beta E_t\left[\frac{(\alpha Z_{t+1}^{1-\alpha} K_{t+1}^{\alpha-1} + 1}{C_{t+1}}\right] = \frac{1}{C_t} \tag{26}$$

In a perfectly competitive environment, the rate of return on capital is

$$\frac{dY_t}{dK_t} = \alpha Z_t^{1-\alpha} K_t^{\alpha-1} = R_t \tag{27}$$

Using (27) in (26) gives the following condition

$$\beta E_t \left[\frac{1 + R_{t+1}}{C_{t+1}} \right] = \frac{1}{C_t} \tag{28}$$

Our objective is to find solutions for output, consumption and investment. We will first compute the steady state solution. In the steady state the values of consumption, income, capital, technology and interest rate are constant, $C_t = C$; $Y_t = Y$; $K_t = K$; $Z_t = Z$ and $R_t = R$. Hence

$$\beta = \frac{1}{1+R} \tag{29}$$

From condition (27) we know that $1 + R = \alpha Z^{1-\alpha} K^{\alpha-1} + 1$. Substituting for $1 + R$ in (29) gives the steady state capital- technology ratio

$$\frac{Z}{K} = \left(\frac{1-\beta}{\alpha\beta} \right)^{\frac{1}{1-\alpha}} \tag{30}$$

Recall that the steady state level of output is $Y = Z^{1-\alpha} K^{\alpha}$. Dividing both sides by K yields the steady state income-capital ratio (which is equal also to the consumption -capital ratio)

$$\frac{Y}{K} = \left(\frac{Z}{K} \right)^{1-\alpha} = \frac{1-\beta}{\alpha\beta} = \frac{C}{K} \tag{31}$$

We will now describe the dynamics of the output, consumption and investment around the steady state. We have two equations (26) and (23) and two unkown, C_t and K_{t+1}:

Let us start with (23) (the budget constraint). We take a first order (linear) Taylor expansion[1] around the steady state.

$$Z_t^{1-\alpha} K_t^{\alpha} - C_t - (K_{t+1} - K_t) \approx Z^{1-\alpha} K^{\alpha} - C + (1-\alpha) \left(\frac{Z_t}{K_t} \right)^{-\alpha} (Z_t - Z) + \left[\alpha \left(\frac{Z_t}{K_t} \right)^{1-\alpha} + 1 \right] (K_t - K) - (\ldots \tag{32}$$

Noting that in the steady state $Z^{1-\alpha} K^{\alpha} - C = 0$ and dividing through this equation by K gives

$$(1-\alpha) \left(\frac{Z_t}{K_t} \right)^{-\alpha} \frac{Z_t - Z}{Z} \frac{Z}{K} + \left[\alpha \left(\frac{Z_t}{K_t} \right)^{1-\alpha} + 1 \right] \frac{K_t - K}{K} - \frac{C_t - C}{C} \frac{C}{K} - \frac{K_{t+1} - K}{K} = 0 \tag{33}$$

[1] A linear approximation of a function $g(x_t)$ around the value $x_t = x$ is simply given by $g(x_t) \approx g(x) + g'(x)(x_t - x)$

An expression such as $\frac{K_t - K}{K}$ represents percentage change in the neighborhood of the steady state. We will denote such changes again by a hatted small letter: $\hat{k}_t$. Note that the ratios $\frac{C}{K}$ and $\frac{Z}{K}$ have already been determined. Using their values from above we get an equation which describes -approximately- capital accumulation:

$$\hat{k}_{t+1} = \frac{1}{\beta}\hat{k}_t + \frac{(1-\alpha)(1-\beta)}{\alpha\beta}\hat{z}_t - \frac{(1-\beta)}{\alpha\beta}\hat{c}_t \tag{34}$$

We need one more equation as we have three unknown variables (output, investment and consumption) and only two equations so far ((??) and (34). The third equation will come from the FOC (28). Taking logs

$$-\log(C_t) = \log(\beta) + \log\left(E_t\left[\frac{1+R_{t+1}}{C_{t+1}}\right]\right) \tag{35}$$

Let us assume that $w_{t+1} = \frac{1+R_{t+1}}{C_{t+1}}$ is log-normally distributed. That is, $\log(w_t) \sim N(\mu;\sigma^2)$ where $\mu = E_t \log(w_t)$ and $\sigma^2 = \text{var}[\log(w_t)]$. Hence $E[w_t] = \exp^{\mu+\frac{1}{2}\sigma^2}$ and

$$E_t\left[\frac{1+R_{t+1}}{C_{t+1}}\right] = \exp E_t[\log(1+R_{t+1}) - \log(C_{t+1})] + \frac{1}{2}\text{var}\left[\log\left(\frac{1+R_{t+1}}{C_{t+1}}\right)\right] \tag{36}$$

The variance is constant by definition. Let us denote it by φ. Substituting (36) into (35) leads to

$$E_t \log(C_{t+1}) - \log(C_t) = E_t \log(1+R_{t+1}) + \log(\beta) + \varphi/2 \tag{37}$$

Let us now express (37) in percentage deviations from the steady state. Subtracting $\log(C)$ in (37) gives

$$E_t[\log(C_{t+1}) - \log(C)] = \log\beta + (\log(C_t) - \log(C)) + E_t[\log(1+R_{t+1})] + \varphi/2 \tag{38}$$

But from (29) we have that $\log\beta = -(1+R)$ so (38) is

$$E_t[\log(C_{t+1}) - \log(C)] = (\log(C_t) - \log(C)) + E_t[\log(1+R_{t+1}) - \log(1+R)] + \varphi/2 \tag{39}$$

An expression such as $\log(C_{t+1}) - \log(C)$ describes percentage deviation from the steady state (that is, it is equal to $(C_{t+1} - C)/C$). Let us again denote such percentage deviations by small letters with a hat. Hence we have

$$E_t[\hat{c}_{t+1}] = \hat{c}_t + E_t\left[\hat{r}_{t+1}\right] + \varphi/2 \tag{40}$$

Recall that $(1 + R_{t+1}) = 1 + \alpha \left(\frac{K_{t+1}}{Z_{t+1}}\right)^{\alpha-1}.$ Taking the total differential of this gives

$$d(1 + R_{t+1}) = \alpha(\alpha - 1)\left(\frac{K_{t+1}}{Z_{t+1}}\right)^{\alpha-2}\frac{dK_{t+1}}{Z_{t+1}} + \alpha(\alpha - 1)\left(\frac{K_{t+1}}{Z_{t+1}}\right)^{\alpha-2}\left(-\frac{K_{t+1}}{Z_{t+1}^2}\right)dZ_{t+1} \tag{41}$$

which in the neighborhood of the steady state takes the form

$$(1 + R)\frac{d(1 + R_{t+1})}{(1 + R)} = \alpha(\alpha - 1)\left(\frac{K}{Z}\right)^{\alpha-1}\frac{dK_{t+1}}{K} - \alpha(\alpha - 1)\left(\frac{K}{Z}\right)^{\alpha-1}\frac{dZ_{t+1}}{Z} \tag{42}$$

Solving the above equation for $\hat{r}_{t+1}$ gives

$$\hat{r}_{t+1} = (\alpha - 1)(1 - \beta)\left(\hat{k}_{t+1} - \hat{z}_{t+1}\right) \tag{43}$$

Substituting (43) into (40) gives

$$E_t[\hat{c}_{t+1}] = \hat{c}_t + (\alpha - 1)(1 - \beta)E_t\left[\hat{k}_{t+1} - \hat{z}_{t+1}\right] + \varphi/2 \tag{44}$$

Together with

$$\hat{k}_{t+1} = \frac{1}{\beta}\hat{k}_t + \frac{(1 - \alpha)(1 - \beta)}{\alpha\beta}\hat{z}_t - \frac{(1 - \beta)}{\alpha\beta}\hat{c}_t \tag{45}$$

we have two equations in two unknowns. We will use the Method of Undetermined coefficients to solve this linear system. Let us assume that the solutions take the form

$$\hat{c}_t = a_{ck}\hat{k}_t + a_{cz}\hat{z}_t \tag{46}$$

$$\hat{k}_{t+1} = a_{kk}\hat{k}_t + a_{kz}\hat{z}_t \tag{47}$$

We will also assume that technology evolves according to an AR(1)-process given by

$$z_{t+1} = \rho z_t + \varepsilon_{t+1} \text{ and } \varepsilon_t \sim N\left(1; \sigma^2\right) \tag{48}$$

Using (46)-(48) to get rid of $\hat{k}_{t+1}$, $\hat{c}_{t+1}$ and $\hat{z}_{t+1}$ in (44) and (45) leads to

$$\hat{c}_t = \Psi_1\hat{k}_t + \Psi_2\hat{z}_t \tag{49}$$

$$\hat{k}_{t+1} = \alpha_3 \hat{k}_t + \alpha_4 \hat{z}_t \tag{50}$$

where the α_i' are functions of the parameters of the model. Now we are almost done. Equations (1.40) and (1.41) imply 4 coefficient restrictions:

$$\begin{aligned} \alpha_1 &= a_{ck} \\ \alpha_2 &= a_{cz} \\ \alpha_3 &= a_{kk} \\ \alpha_4 &= a_{kz} \end{aligned} \tag{51}$$

1.1.3 Numerical solutions

1. A SIMPLE, ONE SECTOR, STOCHASTIC GROWTH MODEL

Economic environment

A single good, a single input (capital), representative agent

Preferences:

$$u(c_t) = \frac{1}{1-\theta} c_t^{1-\theta} \tag{52}$$

Production:

$$f(k_t) = z_t k_t^{\alpha} \tag{53}$$

Physical capital depreciates at the rate of δ per period

Stochastic technology:

$$z_{t+1} = z_t^{\rho} v_{t+1} \qquad v_{t+1} \text{ is iid} \tag{54}$$

Information: v_t becomes known in period t

In the absence of distortions there is an equivalence between the solution to the problem faced by the social planner and the solution obtained in a competitive equilibrium. It saves on notation to use a fictitious social planner who maximizes the utility of a representative agent subject to the resource constraint of the economy

In particular, the value function is

$$V(k_t; z_t) = \max\{u(c_t) + \beta E_t V(k_{t+1}; z_{t+1}) \tag{55}$$

where E_t is conditional expectation taken in period t and k_t and z_t are the state variables: The resource constraint is

$$f(k_t) = c_t + I_t \tag{56}$$

The capital stock evolves as follows

$$k_{t+1} = (1-\delta)k_t + I_t \tag{57}$$

Combining (56) and (57) we have

$$k_{t+1} = (1 - \delta)k_t + f(k_t) - c_t \tag{58}$$

The objective is to maximize (55) subject to (58) and (54) by selecting the optimal sequence $\{c_t; k_{t+1}\}_{t=0}^{\infty}$. Let λ_t be the Langrange multiplier associated with (58). The FOC (Euler equations) are

$$u_{ct} = \lambda_t \tag{59}$$

$$\beta E_t V_{kt+1} = \lambda_t \tag{60}$$

Noting that $V_{kt+1} = \lambda_{t+1}(f_{kt+1} + 1 - \delta)$ and combining (59) and (60) leads to

$$\beta E_t u_{ct+1}(f_{kt+1} + 1 - \delta) = u_{ct} \tag{61}$$

Equations (54), (58) and (61) describe the dynamics of the system. Substituting from (52) and (53) we arrive at a system of three nonlinear, stochastic first order difference equations. In order to study this system we need to consider a tractable, approximate version of this system[2]. There exist several ways of carrying out this approximation (grid, projection method...) which will be discussed later. For the time being we will restrict attention to a simple linear approximation around the deterministic steady state of the model. Let us write (54), (58) and (61) as

$$g_0(z_t; z_{t+1}; v_{t+1}) = 0 \tag{62}$$

$$g_1(c_t; k_t; k_{t+1}; z_t) = 0 \tag{63}$$

$$E_t g_2(c_t; c_{t+1}; k_{t+1}; z_{t+1}) = 0 \tag{64}$$

The first thing we need to do is to calculate the deterministic steady state of the system. That is the values $k = k_t = k_{t+1}$ and $c = c_t = c_{t+1}$ with $z_t = z_{t+1} = z = 1$ that satisfy (62), (63) and (64). It can be easily verified that

$$k = \left(\frac{\beta^{-1} + \delta - 1}{\alpha}\right)^{\frac{1}{\alpha - 1}}$$

and

$$c = (k)^{\alpha} - \delta k$$

Taking a "percentage" first order Taylor approximation of (62), (63) and (64) around $k; c$ and z we have

[2] Another way of dealing with non-linearity is to do the approximation before taking the FOCs. Either start with (1) and (2) but proceed to approximate them by a quadratic polynomial for (1) and a linear function for (2). Or assume from the begining a quadratic utility and linear production functions. I either case the FOC are linear.

$$z'_{t+1} = \rho z'_t + v'_{t+1} \tag{65}$$

$$A1c'_t + A2k'_t + A3k'_{t+1} + A4z'_t = 0 \tag{66}$$

$$B1c'_t + E_tB2c'_{t+1} + B3k'_{t+1} + E_tB4z'_{t+1} = 0 \tag{67}$$

where the A's and B's are functions of $a; b; d; q; r; k; c; z$ and thus known numbers, and $c'_t = (c_t - c)/c$ and so on. Equations (65,66,67) form a system of three stochastic, first order, linear difference equations.

The agents only need information on k_t and z_t in order to decide how much to consume and invest in period t. k and z are the state variables of the system (the former is endogenous while the latter is exogenous). It is then reasonable to assume that their decisions take the form

$$k'_{t+1} = h_1(k'_t; z'_t) \qquad c'_t = h_2(k'_t; z'_t)$$

Our objective is to determine the functions h_1 and h_2. There are several ways of doing so and hence for solving for the variables of interest, c'_t and k'_{t+1}.

I. The method of undetermined coefficients

This is a simple model which, however, may become infeasible if the system is too large because it may require the simultaneous solution of many high order polynomials. For fairly small systems it usually works fine (sometimes one may have to use some tricks to reduce the order of the system to be solved).

Let us assume that

$$h_1(k'_t; z'_t) = e1k'_t + e2z'_t \tag{68}$$

$$h_2(k'_t; z'_t) = e3k'_t + e4z'_t \tag{69}$$

(there is no constant term because everything is measured as deviations from the steady state). Substituting (68) and (69) into (66) and (67) and noting that $E_tc'_{t+1} = E_th_2(t+1)$ and $E_tz'_{t+1} = \rho z'_t$ we arrive at

$$M1k'_t + M2z'_t = 0 \tag{70}$$

$$M3k'_t + M4z'_t = 0 \tag{71}$$

where the M_j s will in general be polynomial functions of the parameters of interest, $e1; e2; e3; e4$. For (70) and (71) to hold for all values of k' and z' it is required that $M1 = M2 = M3 = M4 = 0$. We thus have four equations in four unknowns that can be solved to determine the policy functions h_1 and h_2.

II. The R. Farmer method

Consider again the linearized system of equations (65,66,67). This system can be written as

$$\begin{pmatrix} r & 0 & 0 \\ A4 & A2 & A1 \\ 0 & 0 & B1 \end{pmatrix} \begin{pmatrix} z'_t \\ k'_t \\ c'_t \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & A3 & 0 \\ B4 & B3 & B2 \end{pmatrix} \begin{pmatrix} z'_{t+1} \\ k'_{t+1} \\ c'_{t+1} \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & B4 & 0 & B2 \end{pmatrix} \begin{pmatrix} v'_{t+1} \\ w'_{zt+1} \\ w'_{kt+1} \\ w'_{ct+1} \end{pmatrix} \tag{72}$$

where $w'_{ct+1} = E_t(c'_{t+1}) - c'_{t+1}$(and so on) or, writing it in a more compact form

$$R1s'_t = R2s'_{t+1} + R3n'_{t+1} \tag{73}$$

If R1 is non singular then we multiply (73) through by $R1^{-1}$ to arrive at

$$s'_t = W1s'_{t+1} + W2n'_{t+1} \tag{74}$$

Vaughan (1970) has developed a method for solving equations such as this one. It involves transforming W1 into Jordan canonical form. In particular, W1 can be written as

$$W1 = CJC^{-1} \tag{75}$$

where J is a diagonal matrix with elements the eigenvalues of W1 and C is the matrix whose columns are the eigenvectors of W1: The roots (eigenvalues) of W1 determine the type of solution. Let m be the number of state ("predetermined") variables in the system (in our case there are two, k and z). If the number of eigenvalues that lie outside the unit circle is equal to m then a unique, rational expectations, stationary solution exists; if it is greater than m then there exist multiple stationary equilibria; and if it is smaller than m then no stationary solution exists. Hence, in our case we need two roots outside and one inside the unit circle to have a unique equilibrium. Suppose that the parameters of the model are such that this is indeed the case.

Let us use (75) in (74), multiply through by C^{-1} and then take expectations. The transformed system now takes the form

$$x_t = JE_tx_{t+1} \tag{76}$$

where the first element (row) of x; say, x1; is simply the product of the first row of C^{-1} times the vector column $(z'_t; k'_t; c'_t)$ and so on. Note that the element (row) of x_t that corresponds to the eigenvalue that lies inside the unit circle has a very interesting property. Namely, it has to be equal to zero because -as can be easily seen by recursively substituting for the expectation in the RHS- the corresponding right hand side is equal to zero. But each element of x defines a linear relationship between $c'_t; k'_t$ and z'_t: Subsequently,we have managed to get a solution for c' in terms of k' and z': Substituting this solution into (66) we get a solution for k'_{t+1}: Voila.

III. The Blanchard-Khan method

This is the same method as that employed by Farmer. The only difference is that it does not take advantage of the restriction imposed by the (forward) stable root on the vector of variables. Substituting equation (65) into (66,67) and making sure that we have the state variables at the top and then the endogenous variables produces the following system

$$\begin{pmatrix} A3 & 0 \\ B3 & B2 \end{pmatrix}\begin{pmatrix} k'_{t+1} \\ E_t c'_{t+1} \end{pmatrix} = \begin{pmatrix} A2 & A1 \\ 0 & B1 \end{pmatrix}\begin{pmatrix} k'_t \\ c'_t \end{pmatrix} + \begin{pmatrix} A4 \\ rB4 \end{pmatrix} z'_t \qquad (77)$$

Or written more compactly -after having multiplied by the inverse of the first matrix on the RHS-

$$\begin{pmatrix} k'_{t+1} \\ E_t c'_{t+1} \end{pmatrix} = P \begin{pmatrix} k'_t \\ c'_t \end{pmatrix} + P1 z'_t \qquad (78)$$

The solution method involves writing P in its canonical form $P = C^{-1}JC$, calculating the eigenvalues and eigenvectors of P; decomposing J and C so that the eigenvalues outside the circle (hopefully one in our case, in order to have a unique solution) appear at the top and then using equation (2) from Blanchard and Khan to compute the solution for c'_t and k'_{t+1}.

A complication arises when the system does not consist of first order difference equations. In such a case the method of Farmer (or Blanchard and Khan or King, Plosser and Rebello) is not directly applicable. One must then first reduce the order of the system by suitable transformations (such as defining lagged values as new variables and so on). Note though that while such reduction may be possible most of the time, it is sometimes infeasible.

For instance, suppose that you solved the maximization problem by substituting (58) (for c_t)and (54) in (55) and then linearized around the steady state. The resulting solution would take the form of a linear second order difference equation

$$Y1k'_t + Y2k'_{t+1} + Y3E_t k'_{t+2} + Y4z'_t = 0 \qquad (79)$$

where Y_i are constants. Equation (79) can be written in the form of equation (77) (or equivalently, (72)) by introducing a new variable $x'_{t+1} = k'_{t+2}$

$$\begin{pmatrix} Y2 & Y3 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} k'_{t+1} \\ E_t x'_{t+1} \end{pmatrix} = \begin{pmatrix} -Y1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} k'_t \\ x'_t \end{pmatrix} + \begin{pmatrix} -Y4 \\ 0 \end{pmatrix} z'_t \qquad (80)$$