Primer on asset pricing

Manuel Wälti

This draft: November 2002

Contents

[1] Obstfeld and Rogoff [9], section 5.1, discuss this model in a small open-economy setting.
[2] The model is taken from Obstfeld and Rogoff [9], section 5.2.

1 Introduction

This primer on asset pricing (or macro finance) is written for undergraduate students of macroeconomics at the University of Bern. It is meant to introduce some basic concepts and models of asset pricing in a handy mixture of intuition and rigorous maths and give references to those liking to know more about the topic.

Section 2 introduces basic ideas. Section 3 discusses intertemporal equilibrium asset pricing models in a closed economy framework. Section 4 extends the complete asset markets model to a world economy. Section 5 deals with testable implications of national and international asset pricing models. The manuscript draws on Obstfeld and Rogoff [9] and Blanchard and Fischer [1], among others. Needless to say, all errors are mine.

2 Preliminaries

This section introduces first - within an intuitive reference stock price model - the idea that the price of a security is equal to its fundamental value, namely the present value of its future dividend stream. In the second half of the section, we explore the consumption choice problem of a single agent whose endowment is stochastic.

2.1 An intuitive reference stock price model

2.1.1 The model

Suppose there is an investor who faces two alternatives to invest her money: either she buys a particular equity on the stock market, holds it for one period, and sells it afterwards, or she buys a riskfree bond with constant nominal return r, holds it for one period and then sells it. For the investor to be indifferent between the two alternatives both must yield the same (expected) return

$$\frac{E_t\left\{p_{t+1}\right\}+E_t\left\{d_{t+1}\right\}}{p_t}=1+r \tag{1}$$

$\left(E_t\left\{p_{t+1}\right\}+E_t\left\{d_{t+1}\right\}\right)/p_t$ equals the expected one period gross return of the equity; p_t is the nominal price and d_t the dividend of this particular asset (also in nominal terms). $E_t\left\{x_{t+1}\right\}=E\left\{x_{t+1}\left|\Omega_t\right.\right\}$ is a function that takes the expected value of the term inside the bracket, based on information available in t.

The basic equation (1) can be rewritten as an *expectational difference equation* in p_t

[3] Taken from Obstfeld and Rogoff [9], pp. 290-292.

$$
\begin{aligned}
E_t\{p_{t+1}\} + E_t\{d_{t+1}\} &= rp_t + p_t \\
p_t &= \frac{1}{(1+r)}\left(E_t\{p_{t+1}\} + E_t\{d_{t+1}\}\right)
\end{aligned}
$$

Let's define $\beta \equiv 1/(1+r)$. We end up with

$$
p_t = \beta\left(E_t\{p_{t+1}\} + E_t\{d_{t+1}\}\right) \tag{2}
$$

2.1.2 Recursive substitution

The expectational difference equation in p_t, equation (2), can be solved forward, making use of the *law of iterated expectations* $E_t E_{t+1} E_{t+2} ... E_{t+j-1}\{x_{t+j}\} = E_t\{x_{t+j}\}$

$$
\begin{aligned}
p_t &= \beta\left(E_t\{p_{t+1}\} + E_t\{d_{t+1}\}\right) \\
p_t &= \beta\left(E_t\{\beta\left(E_{t+1}\{p_{t+2}\} + E_{t+1}\{d_{t+2}\}\right)\} + E_t\{d_{t+1}\}\right) \\
p_t &= \beta E_t\{d_{t+1}\} + \beta^2 E_t\{p_{t+2} + d_{t+2}\} \\
p_t &= \beta E_t\{d_{t+1}\} + \beta^2 E_t\{\beta\left(E_{t+2}\{p_{t+3}\} + E_{t+2}\{d_{t+3}\}\right) + d_{t+2}\} \\
p_t &= \beta E_t\{d_{t+1}\} + \beta^2 E_t\{d_{t+2}\} + \beta^3 E_t\{p_{t+3} + d_{t+3}\} \\
p_t &= ... \\
p_t &= \sum_{j=1}^{J} \beta^j E_t\{d_{t+j}\} + \beta^J E_t\{p_{t+J}\}
\end{aligned}
$$

where $\beta^J E_t\{p_{t+J}\}$ denotes the *bubble term*.

Under the appropriate assumptions, $\sum_{j=1}^{\infty} \beta^j E_t\{d_{t+j}\}$ is well behaved and $\lim_{J\to\infty} \beta^J E_t\{p_{t+J}\} = 0$. It follows that the equity price in period t equals the sum of the expected discounted dividends[4]

$$
p_t = \sum_{j=1}^{\infty} \beta^j E_t\{d_{t+j}\} = E_t\left\{\sum_{j=1}^{\infty} \beta^j d_{t+j}\right\} \tag{3}
$$

Digression: The "appropriate" assumptions are:

- $E_t\{d_{t+j}\} \leq h_t \gamma^j$
- $|\beta\gamma| < 1$
- $E_t\{p_{t+j}\}$ does not grow "too fast"

Since $E_t\{d_{t+j}\} \leq h_t \gamma^j$, we have

[4] Note that the expectation of a sum equals the sum of the expectations

$$
E\left\{\sum X_i\right\} = \sum E\{X_i\}
$$

$$
\begin{aligned}
p_t &\leq \sum_{j=1}^{J} \beta^j h_t \gamma^j \\
p_t &\leq h_t \sum_{j=1}^{J} \beta^j \gamma^j
\end{aligned}
$$

As long as $|\beta\gamma| < 1$

$$
\begin{aligned}
\lim_{J \to \infty} \sum_{j=1}^{J} (\beta\gamma)^j &= \frac{1}{1-\beta\gamma} - 1 \\
\lim_{J \to \infty} \sum_{j=1}^{J} (\beta\gamma)^j &= \frac{\beta\gamma}{1-\beta\gamma}
\end{aligned}
$$

so that we can say

$$
\lim_{J \to \infty} p_t = \sum_{j=1}^{J} \beta^j E_t \{d_{t+j}\} \leq \frac{\beta\gamma}{1-\beta\gamma} h_t
$$

Finally, $\lim_{J \to \infty} \beta^J E_t \{p_{t+J}\} = 0$ follows directly from the third condition.

2.1.3 Digression: Test of the efficient markets hypothesis

Consider a variant of equation (1)

$$
\frac{E_t \{p_{j,t+1}\} + E_t \{d_{j,t+1}\}}{p_{j,t}} = 1 + r \tag{4}
$$

where the subscript j stands for any particular share. Note that if this relationship fails, speculators can expect to make a pure profit on their trades in the market for security j.

Under rational expectations, equation (4) can be rewritten as

$$
\frac{p_{j,t+1} + d_{j,t+1}}{p_{j,t}} = 1 + r + \varepsilon_{t+1} \tag{5}
$$

where ε_{t+1} denotes a forecast (or expectational) error, $E_t \{x_{t+1}\} - x_{t+1}$, with $E_t \{\varepsilon_{t+1}\} = 0$; ε_{t+1} is uncorrelated with information available at time t.

Let us define the one period gross return of share j as

$$
1 + r_{j,t+1} \equiv \frac{d_{j,t+1} + p_{j,t+1}}{p_{j,t}}
$$

With this at our disposal equation (5) can be rewritten as

$$
1 + r_{j,t+1} = 1 + r + \varepsilon_{t+1}
$$

From this we can derive the following, testable expression

$$(r_{j,t+1} - r) = \gamma x_t + \epsilon_{t+1}$$

where x_t denotes any elements of the information set upon which $E_t\{\cdot\}$ is based. The null hypothesis of *efficient markets*[5] is that $\gamma = 0$; if not, forecasters would be ignoring information that should be useful in predicting.

2.2 Consumption decision under uncertainty

In the following we describe the optimization problem of a single agent whose future consumption prospects are uncertain (i.e., whose endowment is stochastic) and who has the opportunity to trade so-called Arrow-securities. First, we consider the simple case of two-periods and two states of nature. We then discuss some aspects of the infinite-horizon, multiple states analogue.

2.2.1 Two-periods, two states[6]

Suppose that there is an agent living in a stochastic environment. This agent consumes a physical good, c, and gets an exogenous income (endowment) in form of the physical good, denoted by y. We assume that the agent lives for two periods.

The agent has known first-period income y_1. From the perspective of date 1, however, output on date 2 is uncertain. Either of two states of nature may occur on date 2. In state s, which occurs with probability $\pi(s) \geq 0$, the agent's output equals $y_2(s)$, $s = 1, 2$.

We assume that there is a market in which the agent can buy or sell contingent claims. These contingent claims have period 2 payoffs that depend on the state of nature. Specifically, suppose that on date 1 the agent can buy or sell securities with the following payoff structure: the owner (seller) of the security receives (pays) 1 unit of output on date 2 if state s occurs then, but receives (pays) nothing in all other states. We call this security the *Arrow security* for state of nature s and assume that there is a competitive market in Arrow securities for every state s.

Of course, we also allow the agent to borrow and lend, that is, to sell and buy noncontingent (or riskless) assets that pay $1 + r$ per unit on date 2 regardless of the state of nature, where r is the riskless real rate of interest. If there exist Arrow securities for every state, however, the bond market is redundant. With only two states, for example, the simultaneous purchase of $1 + r$ state 1 Arrow securities and $1 + r$ state 2 Arrow securities assures a pay-off of $1 + r$ output units next period regardless of the economy's state, just as a bond does.

How does an individual evaluate lifetime utility when future consumption prospects are uncertain? An individual with uncertain future income cannot predict his future optimal consumption level exactly. In general, the best he can do is predict a range of consumption levels, each contingent on

[5] The efficient markets hypothesis implies that no investment strategy based on the use of publicly available information can successfully beat the market.

[6] Obstfeld and Rogoff [9], section 5.1, discuss this model in a small open-economy setting.

the state of nature that may occur. In analyzing how an individual plans consumption under uncertainty, we therefore focus on a set of desired contingency plans for consumption. Exactly which plan is brought into play on a future date will depend on the history of economic outcomes up to then.

In terms of our simple two-period, two-state example, we denote by $c_2(s)$, $s = 1, 2$, the two state-contingent consumption plans for date 2. If $s = 1$ in period 2, then the individual will consume $c_2(1)$ and $c_2(2)$ will not be implemented. Conversely, if $s = 2$ in period 2, then the individual will consume $c_2(2)$ and $c_2(1)$ will not be implemented.

How does an individual evaluate lifetime utility when future consumption prospects are uncertain? Our usual assumption will be that satisfaction is measured on date 1 by *lifetime expected utility*, that is, by average lifetime utility given the chosen contingency plans for future consumption. Let c_1 denote consumption on date 1, which must be chosen before the uncertainty is resolved and thus cannot depend on the state of nature that occurs on date 2. The agent's lifetime expected utility on date 1 is

$$\begin{aligned} u_1 &= u(c_1) + \pi(1)\beta u[c_2(1)] + \pi(2)\beta u[c_2(2)] \\ u_1 &= u(C_1) + \sum_{s=1}^{2} \pi(s)\beta u[c_2(s)] \end{aligned}$$

The function $u(\cdot)$ is assumed to be a *von Neumann-Morgenstern utility function*[7] that is increasing, twice continuously differentiable, and strictly concave. It is often specified to

$$u(c_t) = \begin{cases} \frac{1}{1-\sigma}\left(c_t^{1-\sigma} - 1\right) & \text{if } \sigma > 0,\, \sigma \neq 1 \\ \ln(c_t) & \text{if } \sigma = 1 \end{cases}$$

For this specification the elasticity of intertemporal substitution between consumption at any two points in time is constant.[8] We refer to this class of utility functions as the *isoelastic* or *constant relative risk aversion* (CRRA) class (since the elasticity of intertemporal substitution is defined as the inverse of the coefficient of relative risk aversion).

On the budget constraint with Arrow securities We now turn to analyzing the agent's budget constraint under uncertainty, given that there is a competitive market in Arrow securities for every state s. Suppose the agent start out with zero assets. Let $b_2(s)$ be the individual's net purchase of state

[7] For an introduction into choice under uncertainty see e.g. Mas-Colell, Whinston, and Green [8], Chapter 6.

[8] The elasticity of intertemporal substitution is given by the *absolute value* of

$$\eta_{t,s} = \frac{d\left[\frac{c_t}{c_s}\right]}{d\left[\frac{u'(c_t)}{u'(c_s)}\right]} \cdot \frac{\left[\frac{u'(c_t)}{u'(c_s)}\right]}{\left[\frac{c_t}{c_s}\right]}$$

Let $z \equiv \left[\frac{c_t}{c_s}\right]^{-\sigma}$ and, thus, $\left[\frac{c_t}{c_s}\right] = z^{-\frac{1}{\sigma}}$. Further note that $u'(c_t) = c_t^{-\sigma}$ and $\left[\frac{u'(c_t)}{u'(c_s)}\right] = \left[\frac{c_t}{c_s}\right]^{-\sigma}$ ($\equiv z$). Hence, $\eta_{t,s} = \frac{d\left[z^{-\frac{1}{\sigma}}\right]}{dz} \cdot \frac{z}{z^{-\frac{1}{\sigma}}} = -\frac{1}{\sigma} z^{-\frac{1}{\sigma}-1} \cdot z \cdot z^{\frac{1}{\sigma}} = -\frac{1}{\sigma}$. The absolute value is $\frac{1}{\sigma}$.

s Arrow securities on date 1. (Thus, $b_2(s)$ is the stock of state s Arrow securities the individual holds at the end of date 1 and the start of date 2.) Let $p(s)/(1+r)$ denote the price, quoted in in terms of date 1 consumption, i.e. of a unit of consumption delivered on date 2 if and only if the state is s. Note that the agent is a price-taker.

Thus, the agent's budget constraint in date 1 is given by

$$\frac{p(1)}{1+r}b_2(1) + \frac{p(2)}{1+r}b_2(2) = y_1 - c_1$$

(Wee need not explicitly consider purchases of bonds because, as we have seen, bonds are redundant given the two Arrow securities available.) When date 2 arrives, the state of nature s is observed, and the agent will be able to consume the sum of its endowment and any payments on its state s contingent assets,

$$c_2(s) = y_2(s) + b_2(s), \qquad s = 1, 2$$

where $y_2(s)$ denotes the uncertain endowment.

The two budget constraints can be combined to an *intertemporal budget constraint*

$$\begin{aligned}
\frac{p(1)}{1+r}b_2(1) + \frac{p(2)}{1+r}b_2(2) &= y_1 - c_1 \\
\frac{p(1)}{1+r}[c_2(1) - y_2(1)] + \frac{p(2)}{1+r}[c_2(2) - y_2(2)] &= y_1 - c_1 \\
\frac{p(1)}{1+r}c_2(1) - \frac{p(1)}{1+r}y_2(1) + \frac{p(2)}{1+r}c_2(2) - \frac{p(2)}{1+r}y_2(2) &= y_1 - c_1 \\
c_1 + \frac{p(1)}{1+r}c_2(1) + \frac{p(2)}{1+r}c_2(2) &= y_1 + \frac{p(1)}{1+r}y_2(1) \\
&\quad + \frac{p(2)}{1+r}y_2(2) \\
c_1 + \sum_s \frac{p(s)}{1+r}c_2(s) &= y_1 + \sum_s \frac{p(s)}{1+r}y_2(s)
\end{aligned}$$

The agent's problem The agent's optimal portfolio allocations maximize expected utility subject to the intertemporal budget constraint. Let's plug the constraint into the objective and replace c_1.

$$\begin{aligned}
u_1 &= u(c_1) + \pi(1)\beta u[c_2(1)] + \pi(2)\beta u[c_2(2)] \\
u_1 &= u\left[y_1 + \frac{p(1)}{1+r}y_2(1) + \frac{p(2)}{1+r}y_2(2) - \frac{p(1)}{1+r}c_2(1) - \frac{p(2)}{1+r}c_2(2)\right] \\
&\quad + \pi(1)\beta u[c_2(1)] + \pi(2)\beta u[c_2(2)] \\
u_1 &= u\left[y_1 + \frac{p(1)}{1+r}[y_2(1) - c_2(1)] + \frac{p(2)}{1+r}[y_2(2) - c_2(2)]\right] \\
&\quad + \pi(1)\beta u[y_2(1) + b_2(1)] + \pi(2)\beta u[y_2(2) + b_2(2)]
\end{aligned}$$

The resulting problem is to maximize over $b_2(1)$ and $b_2(2)$ the unconstrained expected utility

$$\max_{b_2(1),b_2(2)} u\left[y_1 - \frac{p(1)}{1+r}b_2(1) - \frac{p(2)}{1+r}b_2(2)\right] + \pi(1)\beta u\left[y_2(1)+b_2(1)\right] + \pi(2)\beta u\left[y_2(2)+b_2(2)\right]$$

The optimality conditions The FOC are

$$\frac{p(s)}{1+r}u'(c_1) = \pi(s)\beta u'\left[c_2(s)\right] \qquad s = 1,2$$

This equation is closely related to the intertemporal Euler equation introduced in summary 2. The left-hand side is the cost, in terms of date 1 marginal utility, of acquiring the Arrow-security for state s. The right-hand side is the expected discounted benefit from having an additional unit of consumption in state s on date 2. In equilibrium these must be equal for each possible state in period 2.

As usual this equation can be rearranged to show that the marginal rate of substitution between c_1 and $c_2(s)$ is equal to the two goods' relative price

$$\frac{\pi(s)\beta u'\left[c_2(s)\right]}{u'(c_1)} = \frac{p(s)}{1+r} \qquad s = 1,2$$

2.2.2 Infinite-horizon, $\mathcal{S}$ states

The two-period, two-state setting incorporates all the elements needed to understand the consumption choice problem under uncertainty. Nevertheless we should briefly discuss the extension of the basic setting to a time horizon of more than two periods (in the limiting case: to an infinite number of periods) and a number of states of more than two.

Since we allow for more than two time periods, the state of nature has to be indexed by time. s_t denotes the state of nature in period t. The history of these states up to and including a date t determines the state of nature the economy occupies on date t. We denote the state of nature the economy occupies on date t by $s^t = [s_t, s_{t-1}, ..., s_0]$.

To illustrate the idea consider Figure 1 where - to simplify matters - we assume that $\mathcal{S} = 2$. After two periods, eight mutually exclusive states of the environment are possible. One such state is e.g. the case where $s_0 = 1$, $s_1 = 2$ and $s_2 = 2$. Accordingly, $s^2 = [1, 2, 2]$. Of course, only one out of these eight possible states actually occurs.

Suppose, our agent can purchase a history-dependent consumption plan $c = \{c_t(s^t)\}_{t=0}^{\infty}$, where $s^t = [s_t, s_{t-1}, ..., s_0]$ stands for the state of nature the economy occupies on date t . His lifetime expected utility is given by

$$U(c) = E_0\left\{\sum_{t=0}^{\infty}\beta^t u(c_t)\right\} \tag{6}$$

Let's take a close look at this expression. $E_0\{\cdot\}$ is the mathematical expectation operator, conditioned on s_0, i.e. a probability-weighted average of all possible future contingency plans, in which probabilities are conditioned on all

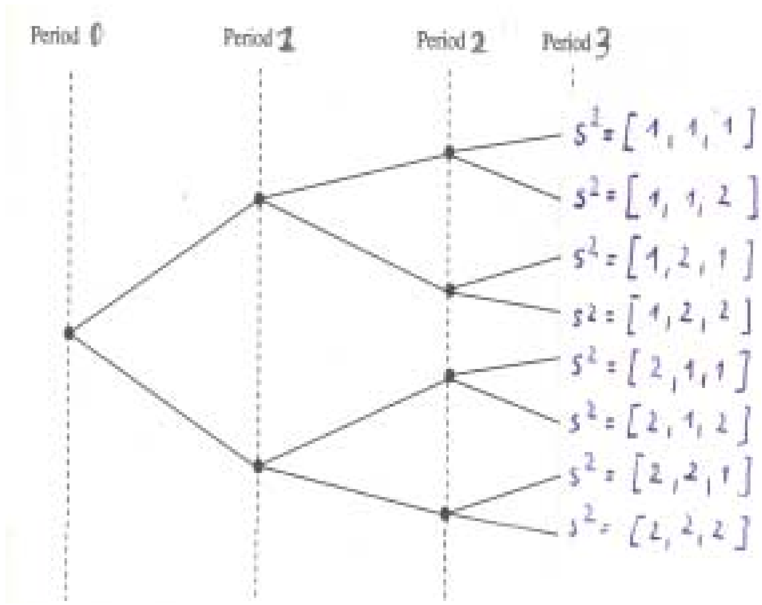

Figure 1: The evolution of uncertainty

information available to the decision maker up to and including date t. To make this evident, let's express it as

$$U(c) = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi\left(s^t \left| s_0 \right.\right) u\left[c_t\left(s^t\right)\right]$$

where $\pi\left(s^t \left| s_0 \right.\right)$ denotes the probability of state s^t conditional on being in state s^0 at date 0. Under the assumption that the stochastic endowment of our agent is *Markov*[9], $\pi\left(s^t \left| s_0 \right.\right)$ does not depend on the history s^{-1}. It is given by

$$\pi\left(s^t \left| s_0 \right.\right) = \pi\left(s_t \left| s_{t-1} \right.\right) \pi\left(s_{t-1} \left| s_{t-2} \right.\right) \cdots \pi\left(s_1 \left| s_0 \right.\right)$$

You remember the (non-stochastic) permanent income model with exogenous interest rates:[10] it illustrates the consumption choice problem of an isolated agent which has access to a bond market. In a deterministic environment the solution to that intertemporal optimization problem would be a sequence of consumption decisions, $\{c_t\}_{t=0}^{\infty}$. These decisions could be made at time zero, since no relevant information is revealed later on. (We say the agent has perfect foresight.)

In a stochastic environment, in contrast, the agent learns over time the realizations of his random endowments. It would be inefficient to ignore this

[9] Just another fancy animal! A stochastic process $\{x_t\}$ is said to have the *Markov property* if for all $k \geq 2$ and all t,

$$\Pr\left(x_{t+1} \left| x_t, x_{t-1}, ..., x_{t-k} \right.\right) = \Pr\left(x_{t+1} \left| x_t \right.\right)$$

[10] If not, you should probably re-read section 3 of summary 2. Summary 2 can be found on the macro II homepage: http://www.vwi.unibe.ch/amakro/Lectures/macroii/macro.htm

information that will be available later on and cast in stone the consumption decisions at time zero. For this reason, the solution to the utility maximization problem is a set of contingency rules (or history-dependent consumption paths), which specify how much to consume at each point in time as a function of the state of the environment in that period: $\{c_t(s^t)\}_{t=0}^{\infty}$. (Note that in the literature s^t is often dropped and we read $\{c_t\}_{t=0}^{\infty}$ although $\{c_t(s^t)\}_{t=0}^{\infty}$ is meant.)

Once we have understood the notation of equation (6) we can set up and solve the household's problem as we did in section 2.2.1. Compare e.g. the manuscript *Dynamic optimization: A toolkit* by the same author, and there the subsection on stochastic dynamic programming.

3 Closed economies

In this section, we look at dynamic stochastic general equilibrium models of closed economies. First, we consider a two-period, $\mathcal{S}$ states, pure exchange economy with stochastic endowments and a *complete contingent-claims market*. That is, there is a single market - that occurs at in period 1 - in which agents trade commodities for future delivery based on the occurrence of future events. The postulation of a contingent commodity for every state in the economy makes the choice of consumption in different states completely analogous to the choice of different consumption goods on a single date.[11] A closely related market structure is the so-called *complete asset market*, which will be discussed next (again, in a two-period setting). In this first subsection, propositions are stated without proofs; the interested student is asked to consult the relevant literature.

Finally, we look at a special case of an infinite horizon, $\mathcal{S}$ states pure exchange economy, namely the representative agent case. The exploration of that model brings to light some very useful results regarding asset pricing.

3.1 A two-period, $\mathcal{S}$ states pure exchange economy

3.1.1 Arrow-Debreu world and Arrow-Debreu equilibrium

Let's consider a pure exchange economy with stochastic endowment. There are I agents, one single physical good (the numéraire), two dates ($t = 1, 2$), and $\mathcal{S}$ states of the world. For convenience we call $s = 0$ the (certain) state of nature in period 1. Further we assume that for all states s there is a market for a state-contingent commodity; what is being purchased (or sold) in the market for the contingent commodity entitles the consumer to receive (or to deliver) amounts of the physical good if and only if state s realizes at date 2. The price of the state-contingent commodity at date 0 is denoted $q(s)$.

In both periods, agent i is endowed with a physical good that yields utility. The endowment in period 1 is certain, while the endowment in period 2 is stochastic and depends on the realization of s. If e.g. in period 2 state $4 < \mathcal{S}$ occurs, agent i gets $\omega_i(4)$.

Agent i sells its state dependent initial endowment, $\omega_i = (\omega_i(0), \omega_i(1), ..., \omega_i(s)) \in IR^{\mathcal{S}+1}$, on the market at the ongoing prices and purchase consumer plan $c_i = (c_i(0), c_i(1), ..., c_i(s)) \in IR^{\mathcal{S}+1}$, where $c_i(s)$ denotes agent i's net purchase of

[11] For an excellent exposition of the general equilibrium theory in a *static* setting compare the script by Georg Müller-Fürstenberg, Department for Applied Microeconomics, Gesellschaftsstrasse 49, 3012 Berne.

the state s contingent commodity on date 1 which entitles agent i to receive $c_i(s)$ of the physical good if and only if state s occurs.

As usual we assume that agents have von Neumann-Morgenstern utility functions. Lifetime expected utility of agent i is given by

$$U_i(c_i) = U_i[c_i(s)] + \beta\left[\sum_s \pi(s) U_i[c_i(s)]\right] \qquad \text{for } i = 1, 2, ..., I$$

The intertemporal (or life-cycle) budget constraint is given by

$$c_i(0) + \sum_s q(s) c_i(s) \leq \omega_i(0) + \sum_s q(s) \omega_i(s)$$

Let's normalize $q(0) = 1$. The consumer problem can be stated as follows

$$\max U_i(c_i)$$

$$\text{s.t.} \quad c_i(0) + \sum_s q(s) c_i(s) \leq \omega_i(0) + \sum_s q(s) \omega_i(s)$$

Definition 1 *An allocation and a system of prices constitute an Arrow-Debreu equilibrium if the consumer problems are solved and the markets are cleared.*

Proposition 2 *If there are complete contingent markets (one per state of the world) then the competitive equilibrium exists and is Pareto optimal.*

3.1.2 Asset markets equilibrium and Arrow-Debreu securities

Contingent markets for each state s exist in reality, but they are rare. What we observe instead are *asset markets*.

Definition 3 *In terms of the two-period setting, an asset n, $n = 1, ..., N$, is described by the amount of the only physical good in our economy it distributes at $t = 2$ as a function of s, $r_n(s)$.*[12] *It's price $q_n(s)$ payable at date 1, is measured in the unit of account at date 1.*

The date 1 payoffs of all N real securities are represented by the payoff matrix

$$R = \begin{pmatrix} r_1(1) & \cdots & r_N(1) \\ \vdots & \ddots & \vdots \\ r_1(\mathcal{S}) & \cdots & r_N(\mathcal{S}) \end{pmatrix}$$

Let's consider an endowment economy with I agents, one single physical good (the numéraire), two dates ($t = 1, 2$) and $\mathcal{S}$ states of the world. The I agents have von Neumann-Morgenstern utility functions $(u_i, ..., u_I)$ and trade N securities with date 2 payoffs given by the matrix R. For the asset market to be *complete*, R must have full rank.

An important example of a complete asset markets economy is the case in which R equals an $N \times N$ identity matrix. In this case the real securities are in

[12] Such an asset is called *real*. If the payoffs of an asset were in paper money, the asset would be called *financial*.

fact *Arrow-Debreu securities*: the nth Arrow security pays off one unit of the single physical good if state s occurs and nothing otherwise.

Let $q = (q_i, ..., q_N)$ denote the vector of security prices where q_n is expressed in the unit of account at date 1 for $n = 1, ..., N$. Let $\theta_i = (\theta_{i1}, ..., \theta_{iN})$ denote the ith agent's portfolio given the number of units of each of the N securities purchased (if $\theta_{in} > 0$) or sold (if $\theta_{in} < 0$). Further suppose that buying and selling these N securities is the only trading opportunity available to agent i.

Lifetime expected utility of agent i is given by

$$U_i(c_i) = U_i[c_i(s)] + \beta\left[\sum_s \pi(s) U_i[c_i(s)]\right] \qquad \text{for } i = 1, ..., I$$

The budget constraint in period 1 (that is, in state 0) is given by

$$c_i(0) + \sum_n q_n \theta_{in} \leq \omega_i(0)$$

where $\omega_i(0)$ denote the initial endowment of agent i in state 0. Finally, the budget constraints in period 2 are given by

$$c_i(s) \leq \omega_i(s) + \sum_n \theta_{in} r_n(s) \qquad \text{for all } s = 1, ..., \mathcal{S}$$

The consumer i's portfolio problem is to choose θ_i such that his expected lifetime utility is maximized subject to the above mentioned set of budget constraints.

Definition 4 *An asset markets equilibrium is a pair of actions and prices* $(\bar{x}, \bar{\theta}, \bar{q})$ *such that the consumer problems are solved and* $\sum_{i=1}^{I} \bar{\theta}_i = 0$

By definition there is a portfolio θ such that $q \cdot \theta < 0$ and $\sum_n \theta_n r_n(s) \geq 0$ for all $s = 1, ..., \mathcal{S}$. We call this an arbitrage opportunity.

Proposition 5 *If an arbitrage opportunity exists, the consumer's portfolio problem does not have any solution.*

Proposition 6 *If there is no arbitrage opportunity there exists a vector of state prices,* $\mu \in IR^{\mathcal{S}}$ *such that*

$$q_n = \sum_s \mu(s) \cdot r_n(s) \qquad \textit{for all } n = 1, ..., N$$

If R *has full rank then the solution* μ *is unique.*

$\mu(s)$ can be interpreted as the present value of one unit of account in state s. That is, an agent would be willing to pay $\mu(s)$ in period 0 to receive 1 unit of account in state s. Thus, $\mu(s)$ is equivalent to the price of the state s contingent commodity in the Arrow-Debreu equilibrium.

You find more about the Arrow-Debreu equilibrium and the asset market equilibrium in a two-period setting in Mas-Colell, Whinston, and Green [8], Chapter 19, in LeRoy and Werner [5], Part 1 to 6, or in Magill and Quinzii [7], Chapter 2.

3.2 A representative agent, infinite horizon, $\mathcal{S}$ states pure exchange economy

A special case of the infinite horizon, pure exchange economy with stochastic endowments is the so-called *Lucas tree model* with identical agents.[13] The model shares features with most of the models used in macroeconomics and asset pricing.

3.2.1 The economy

The model economy is given as follows:[14]

- There is a large number of identical, infinitely lived agents each of whom maximizes lifetime expected utility.
- There is an equal number of trees, $i = 1, ..., N$. Each agent starts life at time zero with one tree. These trees are the only assets in the economy.[15] They are perfectly storable.
- At the beginning of period t, each tree i yields a stochastic dividend (or fruit) in the amount $d_{i,t}$ to its owner. The distribution of $d_{i,t}$ is Markov[16]. This distribution is identical for all trees. The process is known by the agents. The fruit is not storable.
- The market in the ownership of trees is perfectly competitive. In the equilibrium, asset prices clear the market. That is, the total stock positions of all agents are equal to the aggregate number of shares.

3.2.2 The representative consumer's problem

Due to the assumption that all agents are identical with respect to both preferences and endowments, we can work with a representative agent. The agents' economic activity in each period consists of trading shares, where a share represents a claim to the future stream of dividends. We let $p_{i,t}$ be the price of tree i in period t (i.e., of a claim to the entire income stream, $\{d_{i,t+\tau}\}_{\tau=0}^{\infty}$), measured in terms of the consumption commodity and $\theta_{i,t}$ the quantity of tree i that the agent holds between t and $t+1$. Our aim is to find $p_{i,t}$ as a function of the stochastic payoffs, $\{d_{i,t+\tau}\}_{\tau=0}^{\infty}$. Stated formally, the representative consumer chooses $\{\theta_{1,t}, \theta_{2,t}, ..., \theta_{N,t}; c_t\}_{t=0}^{\infty}$ to maximize

[13] Lucas, R. E. Jr. (1978), Asset prices in an exchange economy, Econometrica 46, pp. 1429-45.

[14] This version of the model loosely follows Blanchard and Fischer [1], pp. 510-512. For alternative expositions compare Sargent [11], chapter 3, and Ljungqvist and Sargent [6], pp. 236-343.

[15] This looks a lot like a problem with incomplete markets, since the agent is assumed to be able to buy and sell a single security, the tree. In the representative agent economy, however, the specification of alternative assets becomes irrelevant since these securities must necessarily remain in zero net supply in equilibrium, as we shall see further below.

[16] A stochastic process $\{x_t\}$ is said to have the *Markov property* if for all $k \geq 2$ and all t,

$$\Pr(x_{t+1} | x_t, x_{t-1}, ..., x_{t-k}) = \Pr(x_{t+1} | x_t)$$

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

where $u(c_t)$ is a concave function and $0 < \beta < 1$, subject to the period-by-period budget constraint

$$\sum_{i=0}^{N} \theta_{i,t+1} p_{i,t} + c_t \leq \sum_{i=0}^{N} \theta_{i,t} d_{i,t} + \sum_{i=0}^{N} \theta_{i,t} p_{i,t}$$

The period-by-period budget constraint has the following interpretation: The ownership of share i at the beginning of period t, $\theta_{i,t}$, entitles the owner to receive dividend i in period t and to have the right to sell the tree at price $p_{i,t}$ in period t. Thus, the agent's source of funds in period t are given by the sum of shares that the agent holds times their respective dividends, $\sum_{i=0}^{N} \theta_{i,t} d_{i,t}$, plus the revenue of the sale of the shares at the prevailing prices, $\sum_{i=0}^{N} \theta_{i,t} p_{i,t}$. On the left hand side of the inequality we have got the agent's use of funds, saying that the proceedings are either consumed or spent for the number of shares the agent wants to hold in period $t+1$. This number of share, of course, has to be bought at the prevailing prices.

3.2.3 Market clearing conditions

For a market equilibrium, the quantities of each tree demanded must equal to the (given) supply, that is, the number of shares for each tree must sum up to one. Since there is only one agent, equilibrium implies that $\theta_{i,t} = 1$ for all i, t. In other words, *there is no other agent in the economy with whom to trade insurance, i.e., there is no trade.* From the budget constraint, this implies that consumption of the representative agent must equal to output, which is the sum of dividends

$$c_t \leq \sum_{i=0}^{N} d_{i,t}$$

As we shall see below, the relevant first-order conditions do nothing other than determine the price of the security i that is just sufficient to cause the agent to want neither to buy nor to sell.

3.2.4 Dynamic optimization

The representative agent's problem is a problem of dynamic optimization. There are at least two alternative ways to solve the problem at hand. (For an introduction to dynamic optimization compare e.g. "Dynamic optimization: A tool kit" by Manuel Wälti, downloadable from the macro II homepage.)

Alternative 1 The consumer's dynamic program is given by

$$V(\theta_{1,t}, ...; d_{1,t}, ..., p_{1,t}, ...) = \max \{u(c_t) + \beta E_t V(\theta_{1,t+1}, ...; d_{1,t+1}, ..., p_{1,t+1} ...)\}$$

subject to

$$\sum_{i=0}^{N} p_{i,t} (\theta_{i,t+1} - \theta_{i,t}) \leq \sum_{i=0}^{N} \theta_{i,t} d_{i,t} - c_t$$

where c_t is a control variable, $\theta_{i,t}$ for $i = 1, ..., N$ are state variables, and $d_{i,t}$ and $p_{i,t}$ (both for $i = 1, ..., N$) are exogenous variables (recall that the consumer acts as a price-taker).[17]

Without loss of generality, assume that there is only one tree. Hence, the Lagrangian is given by

$$\begin{aligned}\mathcal{L} &= u(c_t) + \beta E_t V(\theta_{t+1}; d_{t+1}, p_{t+1}) \\ &\quad + \lambda_t \left[\frac{\theta_t d_t}{p_t} - \frac{c_t}{p_t} + \theta_t - \theta_{t+1} \right]\end{aligned}$$

The FOC are

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial c_t} = 0 &= \frac{\partial u(c_t)}{\partial c_t} + \lambda_t \left(\frac{1}{p_t} \right) \\ \frac{\partial \mathcal{L}}{\partial \theta_{t+1}} = 0 &= -\lambda_t + \beta E_t \left\{ \lambda_{t+1} \left(\frac{d_{t+1}}{p_{t+1}} + 1 \right) \right\}\end{aligned}$$

Combining the two FOC yields a version of the well-known Euler equation

$$\begin{aligned}p_t \frac{\partial u(c_t)}{\partial c_t} &= \beta E_t \left\{ \lambda_{t+1} \left(\frac{d_{t+1}}{p_{t+1}} + 1 \right) \right\} \\ p_t \frac{\partial u(c_t)}{\partial c_t} &= \beta E_t \left\{ \frac{\lambda_{t+1}}{p_{t+1}} (d_{t+1} + p_{t+1}) \right\} \\ p_t \frac{\partial u(c_t)}{\partial c_t} &= \beta E_t \left\{ \frac{\partial u(c_{t+1})}{\partial c_{t+1}} (d_{t+1} + p_{t+1}) \right\}\end{aligned}$$

or, alternatively

$$u'(c_t) p_t = \beta E_t \{u'(c_{t+1})(d_{t+1} + p_{t+1})\}$$

It says that in equilibrium the marginal utility cost of the purchase of a security equals the marginal utility gain in the following period.

Alternative 2 Without loss of generality, assume that there is only one tree. Substituting for c_t in the lifetime utility function yields an unconstrained maximization problem

$$\max_{\theta_t} E_0 \left\{ \sum_{t=0}^{\infty} \beta^t u [\theta_t (d_t + p_t) - \theta_{t+1} p_t] \right\}$$

The FOC is

$$\beta^{t-1} E_0 \{u'(c_{t-1})(-p_{t-1})\} + \beta^t E_0 \{u'(c_t)(d_t + p_t)\} = 0$$

which can alternatively be written as

$$u'(c_t) p_t = \beta E_t \{u'(c_{t+1})(d_{t+1} + p_{t+1})\}$$

[17] You might wonder why the consumer should choose $\theta_{i,t}$ although we already know that $\theta_{i,t} = 1$ for all i, t. Recall that the conditions $\theta_{i,t} = 1$ for all i are market clearing conditions. The consumer, of course, acts as if she could freely choose $\theta_{i,t}$ to maximize lifetime utility subject to her budget constraint.

To see this note the following: Taking expectations conditional on information at time $t-2$ of both sides of

$$u'(c_{t-1})\,p_{t-1} = \beta E_{t-1}\{u'(c_t)(d_t + p_t)\}$$

and multiplying by β yields

$$\begin{aligned}
\beta E_{t-2}\{u'(c_{t-1})\,p_{t-1}\} &= \beta E_{t-2}\{\beta E_{t-1}\{u'(c_t)(d_t + p_t)\}\} \\
\beta E_{t-2}\{u'(c_{t-1})\,p_{t-1}\} &= \beta^2 E_{t-2}\{u'(c_t)(d_t + p_t)\}
\end{aligned}$$

Taking expectations conditional on information at time $t-3$ yields

$$\begin{aligned}
\beta E_{t-3}\{\beta E_{t-2}\{u'(c_{t-1})\,p_{t-1}\}\} &= \beta E_{t-3}\left\{\beta^2 E_{t-2}\{u'(c_t)(d_t + p_t)\}\right\} \\
\beta^2 E_{t-3}\{u'(c_{t-1})\,p_{t-1}\} &= \beta^3 E_{t-3}\{u'(c_t)(d_t + p_t)\}
\end{aligned}$$

and so on. Finally, we end up with

$$\beta^{t-1} E_0\{u'(c_{t-1})\,p_{t-1}\} = \beta^t E_0\{u'(c_t)(d_t + p_t)\}$$

3.2.5 Intertemporal Euler equation

The intertemporal Euler equation can be written as an *expectational difference equation* in p_t

$$p_t = E_t\left\{\frac{\beta u'(c_{t+1})}{u'(c_t)}(d_{t+1} + p_{t+1})\right\}$$

p_t is the price at t of an asset with one period payoff $d_{t+1} + p_{t+1}$. $\frac{\beta u'(c_{t+1})}{u'(c_t)}$ is sometimes called the *stochastic discount factor*.

In terms of the one period gross return of an asset, $(d_{t+1} + p_{t+1})/p_t \equiv 1 + r_{t+1}$, we have

$$1 = \beta E_t\left\{\frac{u'(c_{t+1})}{u'(c_t)}(1 + r_{t+1})\right\}$$

Since $E(XY) = E(X)E(Y) + Cov(X, Y)$, the above equation can be rewritten as

$$1 = \beta E_t\left\{\frac{u'(c_{t+1})}{u'(c_t)}\right\} E_t(1 + r_{t+1}) + \beta Cov_t\left\{\frac{u'(c_{t+1})}{u'(c_t)}, (1 + r_{t+1})\right\}$$

or, equivalently,

$$p_t = E_t\left\{\frac{\beta u'(c_{t+1})}{u'(c_t)}\right\} E_t\{(d_{t+1} + p_{t+1})\} + Cov_t\left\{\frac{\beta u'(c_{t+1})}{u'(c_t)}, (d_{t+1} + p_{t+1})\right\}$$

3.2.6 Digression: The consumption-based CAPM

Consider a riskless bond which yields constant gross return, $1 + r$, between t and $t + 1$. The Euler equation becomes to

$$\begin{aligned}
1 &= \beta E_t\left\{\frac{u'(c_{t+1})}{u'(c_t)}(1 + r)\right\} \\
1 &= \beta E_t\left\{\frac{u'(c_{t+1})}{u'(c_t)}\right\}(1 + r) + \beta Cov_t\left\{\frac{u'(c_{t+1})}{u'(c_t)}, (1 + r)\right\}
\end{aligned}$$

Since, obviously, $\frac{u'(c_{t+1})}{u'(c_t)}$ and $(1+r)$ aren't correlated the household's efficiency condition for a riskless asset becomes to

$$1 = \beta E_t \left\{ \frac{u'(c_{t+1})}{u'(c_t)} \right\} (1+r)$$

For the reader's convenience we reproduce here the household's investment efficieny condition for a risky asset (in terms of the one period gross return)

$$1 = \beta E_t \left\{ \frac{u'(c_{t+1})}{u'(c_t)} \right\} E_t \{1 + r_{t+1}\} + \beta Cov_t \left\{ \frac{u'(c_{t+1})}{u'(c_t)}, (1 + r_{t+1}) \right\}$$

Dividing the second through the first gives

$$\begin{aligned}
1 &= E_t \{1 + r_{t+1}\} (1+r)^{-1} \\
&\quad + Cov_t \left\{ \frac{u'(c_{t+1})}{u'(c_t)}, (1 + r_{t+1}) \right\} E_t \left\{ \frac{u'(c_{t+1})}{u'(c_t)} \right\}^{-1} (1+r)^{-1} \\
(1+r) &= 1 + E_t \{r_{t+1}\} + Cov_t \left\{ \frac{u'(c_{t+1})}{u'(c_t)}, (1 + r_{t+1}) \right\} E_t \left\{ \frac{u'(c_{t+1})}{u'(c_t)} \right\}^{-1} \\
E_t \{r_{t+1}\} - r &= -Cov_t \left\{ \frac{u'(c_{t+1})}{u'(c_t)}, (1 + r_{t+1}) \right\} E_t \left\{ \frac{u'(c_{t+1})}{u'(c_t)} \right\}^{-1}
\end{aligned}$$

Recall that

$$E_t \left\{ \frac{u'(c_{t+1})}{u'(c_t)} \right\}^{-1} = \beta (1+r)$$

Thus,

$$E_t (r_{t+1}) - r = -(1+r) Cov_t \left\{ \frac{\beta u'(c_{t+1})}{u'(c_t)}, (1 + r_{t+1}) \right\}$$

The resulting equation is called the *consumption-based CAPM*. It says that the expected premium an asset must yield over the riskless rate of return (called the *asset's risk premium* or *excess return*) depends *negatively* on the covariance of the asset's gross return with the rate of growth of the marginal utility of consumption.

3.2.7 Asset pricing formula

To derive an asset pricing formula the household's investment efficiency condition, which, as we noted above, represents an expectational difference equation in p_t, can be solved forward, again making use of the law of iterated expectations, coupled with the condition $\lim_{T \to \infty} E_t \left\{ \beta^T \frac{u'(c_{t+T})}{u'(c_t)} p_{t+T} \right\} = 0$

$$
\begin{aligned}
p_t &= \beta E_t \left\{ \frac{u'(c_{t+1})}{u'(c_t)} (p_{t+1} + d_{t+1}) \right\} \\
p_t &= \beta E_t \left\{ \frac{u'(c_{t+1})}{u'(c_t)} \left[\left(\beta E_{t+1} \left\{ \frac{u'(c_{t+2})}{u'(c_{t+1})} (p_{t+2} + d_{t+2}) \right\} \right) + d_{t+1} \right] \right\} \\
p_t &= \beta E_t \left\{ \frac{u'(c_{t+1})}{u'(c_t)} d_{t+1} \right\} + \beta E_t \left\{ \frac{u'(c_{t+1})}{u'(c_t)} \beta E_{t+1} \left\{ \frac{u'(c_{t+2})}{u'(c_{t+1})} (p_{t+2} + d_{t+2}) \right\} \right\} \\
p_t &= \beta E_t \left\{ \frac{u'(c_{t+1})}{u'(c_t)} d_{t+1} \right\} + \beta^2 E_t \left\{ \frac{u'(c_{t+1})}{u'(c_t)} \frac{u'(c_{t+2})}{u'(c_{t+1})} (p_{t+2} + d_{t+2}) \right\} \\
p_t &= \beta E_t \left\{ \frac{u'(c_{t+1})}{u'(c_t)} d_{t+1} \right\} + \beta^2 E_t \left\{ \frac{u'(c_{t+1})}{u'(c_t)} \frac{u'(c_{t+2})}{u'(c_{t+1})} d_{t+2} \right\} \\
&\quad + \beta^2 E_t \frac{u'(c_{t+1})}{u'(c_t)} \frac{u'(c_{t+2})}{u'(c_{t+1})} p_{t+2}
\end{aligned}
$$

and so on until we get

$$
\begin{aligned}
p_t &= E_t \left\{ \sum_{j=1}^{\infty} \beta^j \left[\prod_{s=0}^{j-1} \frac{u'(c_{t+s+1})}{u'(c_{t+s})} d_{t+j} \right] \right\} \\
p_t &= E_t \left\{ \sum_{j=1}^{\infty} \beta^j \frac{u'(c_{t+j})}{u'(c_t)} d_{t+j} \right\}
\end{aligned}
$$

Since $E(\sum Z) = \sum E(Z)$ and $E(XY) = E(X)E(Y) + Cov(X,Y)$, this can alternatively be written as

$$
p_t = \sum_{j=1}^{\infty} E_t \left\{ \frac{\beta^j u'(c_{t+j})}{u'(c_t)} \right\} E_t \{d_{t+j}\} + \sum_{j=1}^{\infty} Cov_t \left\{ \frac{\beta^j u'(c_{t+j})}{u'(c_t)}, d_{t+j} \right\} \tag{7}
$$

where $E_t \left\{ \beta^j u'(c_{t+j}) / u'(c_t) \right\}$ is the date t market discount factor for noncontingent date $t+j$, $j > t$, consumption, that is, the inverse of the gross interest rate on a riskless j period discount bond.

The pricing formula states that an asset's price is the expected present value of payouts (discounted at riskless rates) *plus* a sum of risk adjustments, each of which reflects the asset's contribution to consumption insurance on a different future date. Other things being equal, an asset that tends to pay off unexpectedly well on date $t+j$ when the marginal utility of consumption is unexpectedly high (meaning that consumption itself is unexpectedly low) has value as a consumption hedge and therefore will command a price above its 'risk neutral' or actuarially fair price. (Compare asset pricing formula (7) with formula (3) of Section 2; what are the differences?

3.2.8 Some illustrative examples

Consider a multiple payout security such as a stock (that is, a claim to the infinite sequence $\{d_{t+\tau}\}_{\tau=0}^{\infty}$, where $d_{t+\tau}$ is random). What is the price of such a security? The asset pricing formula gives us the answer:

$$p_t^d = \sum_{j=1}^{\infty} \beta^j E_t \left\{ \frac{u'(c_{t+j})}{u'(c_t)} d_{t+j} \right\}$$

Now assume that the agent is risk neutral. [18] The pricing formula becomes

$$p_t^d = \sum_{j=1}^{\infty} \beta^j E_t \{d_{t+j}\}$$

which is the pricing equation (3) from section 2.

Next, consider an asset with a stochastic single period payout $\delta_{t+\tau}$ (in other words, in period τ the payoff is random; in period $s \neq \tau$ the payoff is zero). What is the price of such a security? Again, the asset pricing formula provides us with the answer:

$$p_t^{\delta_\tau} = \beta^\tau E_t \left\{ \frac{u'(c_{t+\tau})}{u'(c_t)} \delta_{t+\tau} \right\}$$

A special case of such an asset is one which yields a sure payoff of one unit of the numéraire in $t+1$. The value of such a claim is

$$p_t^b = \beta E_t \left\{ \frac{u'(c_{t+1})}{u'(c_t)} \cdot 1 \right\}$$

This price is usually defined as $1/(1+r)$. Given this definition, we can write

$$\begin{aligned} \frac{1}{1+r} &= \beta E_t \left\{ \frac{u'(c_{t+1})}{u'(c_t)} \right\} \\ 1 &= \beta E_t \left\{ \frac{u'(c_{t+1})}{u'(c_t)} (1+r) \right\} \end{aligned}$$

Finally, consider an asset which pays exactly one unit of the numéraire in state s_j in period τ and zero otherwise (that is, in all but state s_j in period τ the payoff is zero). The value of this claim is given by

$$p_t^{d_\tau} = \beta^\tau \pi(s^\tau | s_t) \frac{u'(c_{t+\tau})}{u'(c_t)}$$

3.2.9 Solving for the general equilibrium

Representative agent asset-pricing models can be constructed by the following steps (compare Ljungqvist and Sargent [6], pp. 236-237):

1. Describe the preferences, technology, and endowments of a dynamic economy, then solve for the equilibrium intertemporal consumption allocation.

[18] An agent with von Neumann-Morgenstern utility function $u(\cdot)$ is risk neutral if

$$E\{u(c)\} = u(E\{c\})$$

- compare e.g. LeRoy and Werner [5], chapter 9. In the present setting where $c_t = d_t$, expected utility is given by a constant, $E\{c\} = c$. It follows that $u'(c)$ is constant.

2. Set up a competitive market in some particular asset that represents a specific claim on future consumption goods. Permit agents to buy and sell at equilibrium asset prices subject to particular borrowing and short-sales constraints. Find an agent's Euler equation for this asset.

3. Equate the consumption that appears in the Euler equation derived in step 2 to the equilibrium consumption derived in step 1. This procedure will give the asset price at t as a function of the state of the economy at t.

In the Lucas tree model, a planner that treats all agents the same would like to maximize $E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$ subject to $c_t = d_t$. Evidently the solution is to set c_t equal to d_t. After substituting this consumption allocation into the above investment efficient condition, we arrive at expressions for the risk-free interest rate and the share price,

$$u'(d_t)(1+r)^{-1} = \beta E_t \{u'(d_{t+1})\}$$

$$u'(d_t) p_t = \beta E_t \{u'(d_{t+1})(d_{t+1} + p_{t+1})\}$$

Notice: The basic asset pricing formula holds whether or not the economy is an exchange economy. For examples of representative agent asset-pricing models of *production economies* compare Rouwenhorst [10] and Jermann [3].

4 World economies

In this section we first extend the complete securities setting to a world economy. Second, we consider particular types of incomplete capital markets models of a world economy.

4.1 Complete markets

4.1.1 A two-period, $\mathcal{S}$ states, two-country endowment economy[19]

We assume that the world economy consists of two countries, Home and Foreign. Production is exogenously given in each country: output levels fluctuate across $\mathcal{S}$ states of nature. There is a worldwide market in which people can buy or sell Arrow securities. The notation is exactly the same as in section 2.3.1 except that capital letters instead of lower case letters are used. $B_2(s)$, for instance, is the Home country's representative individual's net purchase of state s Arrow securities on date 1.

Home country's representative agent's life-time utility is given by

$$u(C_1) + \sum_s \pi(s) \beta u[C_2(s)]$$

The Home's intertemporal budget constraint is represented by

$$Y_1 + \sum_s \frac{p(s)}{1+r} Y_2(s) = C_1 + \sum_s \frac{p(s)}{1+r} C_2(s)$$

[19] The model is taken from Obstfeld and Rogoff [9], section 5.2.

Home's necessary Euler equations are of the same form as for the individual agent in section 2.3.1, with due allowance for the possibility that $\mathcal{S} > 2$

$$\frac{p(s)}{1+r}u'(C_1) = \pi(s)\beta u'[C_2(s)] \qquad s = 1, 2, ..., \mathcal{S}$$

Analogous conditions hold for the Foreign country; foreign quantities corresponding to Home's are marked with an asterisk $(*)$.

$$\frac{p(s)}{1+r}u'(C_1^*) = \pi(s)\beta u'[C_2^*(s)] \qquad s = 1, 2, ..., \mathcal{S}$$

Home and Foreign consumers have the same degree of risk aversion.

Global general equilibrium conditions requires that supply and demand balance in $\mathcal{S}+1$ markets: the market for date 1 output and those for date 2 output delivered in each of the $\mathcal{S}$ states of nature

$$\begin{aligned} C_1 + C_1^* &= Y_1 + Y_1^* \\ C_2(s) + C_2^*(s) &= Y_2(s) + Y_2^*(s) \qquad s = 1, 2, ..., \mathcal{S} \end{aligned}$$

Implications for marginal utilities The complete-markets model has strong implications concerning correlations in international consumption levels across time and across states of nature. These strong predictions arise because complete markets allow all individuals in Home and Foreign to equate their marginal rates of substitution between current consumption and state-contingent future consumption to the same state-contingent security prices.

Home's and Foreign's necessary Euler equations imply (by the law of one price)

$$\frac{\pi(s)\beta u'[C_2(s)]}{u'(C_1)} = \frac{p(s)}{1+r} = \frac{\pi(s)\beta u'[C_2^*(s)]}{u'(C_1^*)} \tag{8}$$

where $p(s)/(1+r)$ denotes the state contingent security price, and

$$\frac{\pi(s)\beta u'[C_2(s)]}{\pi(s')\beta u'[C_2(s')]} = \frac{p(s)}{p(s')} = \frac{\pi(s)\beta u'[C_2^*(s)]}{\pi(s')\beta u'[C_2^*(s')]} \tag{9}$$

for all states s and s', where $p(s)/p(s')$ denotes the relative price of state s in terms of state s'. All individuals' marginal rates of substitution in consumption - *over time and across states* - are equal, so no potential gains from trade remain to be exploited.

Implications for consumption levels Equations (8) and (9) relate only marginal utilities of consumption. But specific utility functions yield implications for consumption levels.

For a common CRRA period utility function

$$u(C_t) = \frac{1}{1-\rho}\left(C_t^{1-\rho} - 1\right), \quad \rho > 0,\ \rho \neq 1$$

the above Euler equations for state s securities show that

$$\begin{aligned} C_2(s) &= \left[\frac{\pi(s)\beta(1+r)}{p(s)}\right]^{1/\rho} C_1 \qquad s = 1, 2, ..., \mathcal{S} \\ C_2^*(s) &= \left[\frac{\pi(s)\beta(1+r)}{p(s)}\right]^{1/\rho} C_1^* \qquad s = 1, 2, ..., \mathcal{S} \end{aligned}$$

These and the equilibrium conditions give together

$$\begin{aligned}
Y_2(s) + Y_2^*(s) &= C_2(s) + C_2^*(s) \qquad s = 1, 2, ..., \mathcal{S} \\
Y_2(s) + Y_2^*(s) &= \left[\frac{\pi(s)\beta(1+r)}{p(s)}\right]^{1/\rho} [C_1 + C_1^*] \qquad s = 1, 2, ..., \mathcal{S} \\
Y_2(s) + Y_2^*(s) &= \left[\frac{\pi(s)\beta(1+r)}{p(s)}\right]^{1/\rho} Y_1^w \qquad s = 1, 2, ..., \mathcal{S}
\end{aligned}$$

which implies that the date 1 price of the state s contingent security is

$$\frac{p(s)}{(1+r)} = \pi(s)\beta\left[\frac{Y_2^w(s)}{Y_1^w}\right]^{-\rho} \qquad s = 1, 2, ..., \mathcal{S}$$

where we define $Y^w \equiv Y + Y^*$ as total world output.

With the above specific momentary utility equations (8) and (9), combined with the equations for state-contingent prices, imply

$$\begin{aligned}
\frac{\pi(s)\beta C_2(s)^{-\rho}}{C_1^{-\rho}} &= \pi(s)\beta\left[\frac{Y_2^w(s)}{Y_1^w}\right]^{-\rho} = \frac{\pi(s)\beta C_2^*(s)^{-\rho}}{C_1^{*-\rho}} \qquad (10) \\
\frac{C_2(s)}{C_1} &= \frac{Y_2^w(s)}{Y_1^w} = \frac{C_2^*(s)}{C_1^*}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\pi(s)\beta C_2(s)^{-\rho}}{\pi(s')\beta C_2(s')^{-\rho}} &= \frac{\pi(s)\beta\left[\frac{Y_2^w(s)}{Y_1^w}\right]^{-\rho}}{\pi(s')\beta\left[\frac{Y_2^w(s')}{Y_1^w}\right]^{-\rho}} = \frac{\pi(s)\beta C_2^*(s)^{-\rho}}{\pi(s')\beta C_2^*(s')^{-\rho}} \qquad (11) \\
\frac{C_2(s)}{C_2(s')} &= \frac{Y_2^w(s)}{Y_2^w(s')} = \frac{C_2^*(s)}{C_2^*(s')}
\end{aligned}$$

for all states.

Equation (11) implies

$$\frac{C_2(s)}{Y_2^w(s)} = \frac{C_2(s')}{Y_2^w(s')} = \mu, \qquad \frac{C_2^*(s)}{Y_2^w(s)} = \frac{C_2^*(s')}{Y_2^w(s')} = 1 - \mu$$

meaning that Home consumption is a constant fraction μ of world date 2 output regardless of the state. (Foreign's state-invariant share, correspondingly, is $1 - \mu$.)

Equation (10) says that consumption growth rates are the same across countries in every state and are equal to the growth rate of world output; it can be rearranged to (for all s)

$$\frac{C_2(s)}{Y_2^w(s)} (= \mu) = \frac{C_1}{Y_1^w}, \qquad \frac{C_2^*(s)}{Y_2^w(s)} (= 1 - \mu) = \frac{C_1^*}{Y_1^w}$$

meaning that the countries' date 1 consumption shares in world output are the same as their date 2 shares.

4.1.2 An infinite horizon, $\mathcal{S}$ states multicountry endowment economy

The two-period model can be extended to encompass an infinite horizon. For a formal discussion see Obstfeld and Rogoff [9], Appendix 5C.

4.2 Departures from the complete Arrow-securities market assumption

Let us briefly discuss two important departures from the complete Arrow-securities market assumption.

One such departure is a world economy in which the only risky assets traded are claims to countries' uncertain outputs. Obstfeld and Rogoff [9], section 5.3, explore a two-period, $\mathcal{S}$ states, multicountry endowment economy in which individuals in each country hold securities that pay out claims against each other's uncertain outputs. Also, they assume that people throughout the world have identical CRRA momentary utility functions. Obstfeld and Rogoff then show that even when the assets traded are limited to riskless (or non-contingent) real bonds *and* shares in each of the N countries' future outputs, the equilibrium allocation is *Pareto efficient*. Indeed, it is identical to the equilibrium reached through trade in a full set of $\mathcal{S}$ Arrow securities, even when $\mathcal{S}$ is much greater than N.[20]

This finding, however, is sharply altered when the *only* internationally traded assets are riskless (or non-contingent) real bonds denominated in a composite consumption good, as it is typically assumed in the *intertemporal approach to the current account* literature.[21] In such a setting, individuals can engage in consumption-smoothing, but not in risk pooling. For a discussion of the effects of productivity shocks in complete-markets versus bonds-only models with infinite horizons compare Obstfeld and Rogoff [9], section 5.2.3 (two-country model; perfect foresight) and Schmitt-Grohé and Uribe [12] (small-open economy models; stochastic setting).

5 Empirical evidence

The complete asset markets models considered so far yield strong empirical predictions. Let us briefly discuss two of them.

5.1 The equity premium puzzle

One of the most striking mysteries in empirical finance is the enormous observed average rate of return differential between common stocks and government securities. Over the past seventy years, the annual rate of return on stocks has exceeded that on governement bonds by more that 5 percent across most major stock markets of the world.

Plainly it is no puzzle that the expected return on stocks is higher than on safer assets. It is the *size* of the differential that perplexes researches. For seemingly reasonable parametrization of a general equilibrium complete markets model, premiums of more than 1 or 2 percent cannot easily be rationalized. This problem is the *equity premium puzzle*, first pointed out by Mehra and Prescott

[20] In the case of identical CRRA period utility functions this result is quite straightforward to derive. However, with other utility functions, or with international differences in preferences (e.g., different constant coefficients of risk aversion), more restrictive assumptions are required to make sure that trade in equities is a perfect substitute for trade in a complete set of Arrow-securitites.

[21] For an introduction into the intertemporal approach to the current account compare Obstfeld and Rogoff [9], chapter 1, 2, and 3.1.

(1985)[22] for annual 1889-1978 United States data. For an illustration of the basic problem compare Obstfeld and Rogoff [9], section 5.4.2. Alternatively, compare Siegel and Thaler [13].

5.2 The home bias puzzle[23]

The opportunities for global risk sharing that the international capital market offers should lead investors to diversify their portfolios over investments in many countries. In practice, however, little of the sort seems to happen. The investors of most countries seem to hold a very large share of their equity wealth at home. The contradiction between the obvious benefits of holding a globally dispersed set of equities and the apparent reluctance to do so is known as the *home bias puzzle*.

One way to state the home bias puzzle is the following: We have seen that when two countries have identical CRRA preferences, their shares of world consumption are constant across time, so that their rates of consumption growth must always be equal ex post. However, even for two countries n and m with different constant coefficients of risk aversion, ρ_n and ρ_m, and different subjective discount factors, β_n and β_m, the model still yields a very strong empirical prediction. If we use lowercase c and y to denote per capita consumption and output, then equation (8) implies

$$\begin{aligned}
\pi(s)\,\beta_n\left[\frac{C_2^n(s)}{C_1^n}\right]^{-\rho_n} &= \pi(s)\,\beta_m\left[\frac{C_2^m(s)}{C_1^m}\right]^{-\rho_m} \\
\frac{\beta_n}{\beta_m}\left[\frac{C_2^n(s)}{C_1^n}\right]^{-\rho_n} &= \left[\frac{C_2^m(s)}{C_1^m}\right]^{-\rho_m} \\
\log\left(\frac{\beta_m}{\beta_n}\right)-\rho_n\log\left(\frac{C_2^n(s)}{C_1^n}\right) &= -\rho_m\log\left(\frac{C_2^m(s)}{C_1^m}\right) \\
\log\left(\frac{C_2^n(s)}{C_1^n}\right) &= \frac{\rho_m}{\rho_n}\log\left(\frac{C_2^m(s)}{C_1^m}\right)+\log\left(\frac{\beta_m}{\beta_n}\right)
\end{aligned}$$

The resulting equation shows that any two countries' ex post consumption growth rates, although individually random, are perfectly statistically correlated: they have a correlation coefficient of 1.[24]

Does this perfect-correlation implication match the data? Embarrassingly for the simplest one-good version of the complete markets model, it does not. Figure 2 (which is taken from Obstfeld and Rogoff [9], p. 291) presents 1973-92 estimates of correlation coefficients between national per capita consumption growth and world per capita consumption growth. These are compared with correlation coefficients between national per capita output growths and world per capita output growth.

[22] Mehra, Rajnish, and Edward C. Prescott (1985), The equity premium: A puzzle, Journal of Monetary Economics 15 (March), 145-161

[23] Taken from Obstfeld and Rogoff [9], pp. 290-292.

[24] To see this note that the correlation coefficient between two random variables X and Y, is defined as

$$\text{Corr}\{X,Y\}=\frac{\text{Cov}\{X,Y\}}{\sqrt{\text{Var}\{X\}}\sqrt{\text{Var}\{X\}}}$$

If X and Y are linearly related, so that $Y=a_0+a_1X$ for some constants a_0 and $a_1>0$, then these random variables are perfectly correlated.

Consumption and Output: Correlations between Domestic and World Growth Rates, 1973–92

Country	Corr $(\hat{c}, \hat{c}^*)$	Corr $(\hat{y}, \hat{y}^*)$
Canada	0.56	0.70
France	0.45	0.60
Germany	0.63	0.70
Italy	0.27	0.51
Japan	0.38	0.46
United Kingdom	0.63	0.62
United States	0.52	0.68
OECD average	0.43	0.52
Developing country average	−0.10	0.05

Note: The numbers Corr$(\hat{c}, \hat{c}^*)$ and Corr$(\hat{y}, \hat{y}^*)$ are the simple correlation coefficients between the annual change in the natural logarithm of a country's real per capita consumption (or output) and the annual change in the natural logarithm of the rest of the world's real per capita consumption (or output), with the "world" defined as the 35 benchmark countries in the Penn World Table (version 5.6). Average correlations are population-weighted averages of individual country correlations. The OECD average excludes Mexico.

Figure 2: The home bias puzzle

For the seven largest industrial countries, the correlation between domestic and world consumption growth is lower in almost every case than the correlation between domestic and world output growth; for developing countries, the average correlation between domestic and world consumption growth is actually slightly negative.

For a more comprehensive discussion of the home bias puzzle as well as a discussion of possible explanations compare Lewis [4], section 3.

References

[1] Blanchard, Olivier Jean, and Stanley Fischer (1989), Lectures on Macroeconomics, The MIT Press

[2] Farmer (1993), The Macroeconomics of Self-Fulfilling Prophecies, MIT Press

[3] Jermann, Urban J. (1998), Asset Pricing in Production Economies, Journal of Monetary Economics 41, 257-275

[4] Lewis, Karen K. (1999), Trying to Explain Home Bias in Equities and Consumption, Journal of Economic Literature, Vol. XXXVII (June)

[5] LeRoy, Stephen F., and Jan Werner (2001), Principles of Financial Economics, Cambridge University Press

[6] Ljungqvist and Sargent (2000), Recursive Macroeconomic Theory, MIT Press

[7] Magill, Michael, and Martine Quinzii (1996), Theory of Incomplete Markets - Vol. 1, MIT Press

[8] Mas-Colell, Andreu, Michael D. Whinston and Jerry R. Green (1995), Microeconomic Theory, Oxford University Press

[9] Obstfeld and Rogoff (1996), Foundations of International Macroeconomics, MIT Press

[10] Rouwenhorst, K. Geert (1995), Asset Pricing Implications of Equilibrium Business Cycle Models, in: Cooley, Thomas F., ed., Frontiers of Business Cycle Research, Princeton University Press

[11] Sargent, Thomas J. (1997), Dynamic Macroeconomic Theory, Harvard University Press

[12] Schmitt-Grohé, Stephanie, and Martin Uribe (2002), Closing small open economy models, CEPR Discussion paper series No. 3096

[13] Siegel, Jeremy J. and Richard. H. Thaler (1997), Anomalies: The Equity Premium Puzzle, Journal of Economic Perspectives, Winter 11(1), 191-200

[14] Stokey, Nancy, and Robert E. Lucas, Jr. (with Edward C. Prescott) (1989), Recursive Methods in Economic Dynamics, Harvard University Press