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FIRST ORDER LINEAR DIFFERENCE EQUATIONS
(see also A. Chiang, Fundamental methods of mathematical economics)

Both difference and differential equations describe the evolution of a variable(s) over time, t; the former in discrete time (t=1,2,3..) and the latter in continuous time.

CASE I. $x(t) - ax(t-1) = b$ (1)

a) Solution of homogeneous part: $x(t) - ax(t-1) = 0$
$x(t) - ax(t-1) = 0 \Rightarrow x(t) = ax(t-1) = a[ax(t-2)]= a[a[ax(t-3)]] = . . . = a^t x(0)$

if $|a| < 1$ then x(t) eventually becomes zero
if $|a| > 1$ then x(t) eventually diverges to infinity

b) Particular solution:

Since the right hand side (RHS) of (1) is a constant let us assume that x(t) = k for all t, where k is a constant to be determined. Substituting in (1) and solving for k produces k = b/(1-a).
The general solution of (1) is the sum of the homogeneous and particular solutions, that is
$x(t) = a^t x(0) + b/(1-a)$. If $|a| < 1$ then x(t) always converges to b/(1-a) no matter what the starting value of x is.

CASE II $x(t) - ax(t-1) = b(t)$ (2)

Now the RHS is a function of time. The solution to the homogeneous part is the same as before but the particular solution will depend on the form of b(t). A common procedure for solving (2) that uses the so called "lag-operator" is as follows: Rewrite (2) as

$(1-aL)x(t) = b(t)$, where L is the lag operator so that $Lx(t) = x(t-1)$, $L^2x(t) = x(t-2)$ and so on
Subsequently,
$x(t) = b(t)[1/(1-aL)]$. Now notice that if $|aL| < 1$ then 1/(1-aL) is the sum of a convergent geometric progression, that is
$1/(1-aL) = 1+aL+(aL)^2+(aL)^3 + ...$
Hence, $b(t)[1/(1-aL)] = b(t)[1+aL+(aL)^2+(aL)^3...] = b(t) + aLb(t) + a^2L^2b(t) + a^3L^3b(t) + ...=$
$b(t) + ab(t-1) + a^2b(t-2) + a^3b(t-3) + ... = \Sigma_{j=0}a^jb(t-j)$

The general solution is then $x(t) = a^t x(0) + \Sigma^t_{j=0}a^jb(t-j)$