Haris Dellas, Fall 1998

A SIMPLE, ONE SECTOR, TWO FACTOR, STOCHASTIC, EXOGENOUS GROWTH MODEL

Economic environment

A single good, two inputs (capital and labor), a representative agent

Preferences:

$$u(C_t, 1 - H_t) = \frac{1}{\psi}[C_t^{1-\gamma}(1 - H_t)^{\gamma}]^{\psi} \quad \psi \neq 1 \tag{1}$$

Production:

$$Y_t = F(K_t, H_t, z_t) = K_t^{\alpha}(Z_t H_t)^{1-\alpha} \tag{2}$$

Physical capital depreciates at the rate of d per period

Technology has a deterministic trend and a stochastic cycle:

$$Z_t = e^{\mu t + \theta_t} \quad \mu > 0; \ \theta_{t+1} = (1 - \rho)\theta + \rho\theta_t + v_{t+1}; \quad v_{t+1} \ \text{iid} \tag{3}$$

Information: v_t becomes known in period t

The value function is

$$V(K_t, Z_t) = \max\{u(C_t, H_t) + bE_t V(K_{t+1}, Z_{t+1})\} \tag{4}$$

where E_t is conditional expectation taken in period t and k_t and z_t are the state variables. The resource constraint is

$$F(K_t, H_t, Z_t) = Y_t = C_t + I_t \tag{5}$$

The capital stock evolves as follows

$$K_{t+1} = (1 - d)K_t + I_t \tag{6}$$

Combining (5) and (6) we have

$$K_{t+1} = (1 - d)K_t + F(K_t, H_t, Z_t) - C_t \tag{7}$$

The objective is to maximize (4) subject to (7) and (3) by selecting the optimal sequence $\{C_t, H_t, K_{t+1}\}_{t=0}^{\infty}$. Let λ_t be the Langrange multiplier associated with (7). The FOC (Euler equations) are

$$u_{Ct} = \lambda_t \tag{8}$$

$$u_{Ht} = \lambda_t F_{Ht} \tag{9}$$

$$bE_t V_{Kt+1} = \lambda_t \tag{10}$$

Noting that $V_{Kt+1} = \lambda_{t+1}(F_{Kt+1} + 1 - d)$ and combining (8) and (10) leads to

$$bE_t u_{Ct+1}(f_{Kt+1} + 1 - d) = u_{Ct} \tag{11}$$

$$u_{Ht} = u_{Ct} F_{Ht} \tag{12}$$

Equations(3), (7), (11) and (12) describe the dynamics of the system. Note that output, consumption and the capital stock are non-stationary (because $\mu > 0$). We can transform them into stationary processes by noting that in a balanced growth steady state Z_t, Y_t, C_t, and K_t all grow at the exogenous rate of μ. Let us use the definitions

$$z_t = Z_t e^{-\mu t} = e^{\theta_t} \quad c_t = C_t e^{-\mu t} \quad k_t = K_t e^{-\mu t} \quad f(k_t, H_t, z_t) = e^{-\mu t} F(K_t, H_t, Z_t) \quad \beta = be^{\mu[\psi(1-\gamma)-1]} \tag{13}$$

Hence, the variables c_t, y_t and k_t are stationary. Using (13) in (3),(11) and (12) we arrive at a system of four nonlinear, stochastic first order difference equations.

$$z_{t+1} = e^{(1-\rho)\theta + \rho\theta_t + v_{t+1}} \tag{14}$$

$$k_{t+1} = (1-d)k_t + f(k_t, H_t, z_t) - c_t \tag{15}$$

$$\beta E_t u_{ct+1}(f_{kt+1} + 1 - d) = u_{ct} \tag{16}$$

$$u_{Ht} = u_{ct} f_{Ht} \tag{17}$$

Again we proceed as in the case of the one factor growth model. We first calculate the steady state solution $\{z, k, c, H\}$ (note that the steady state value of $\theta_t = \theta$ and thus $z = e^{\theta}$).

Let $\{\theta, \alpha, \gamma, \psi, \rho, \mu, b, d\} = \{1, 0.35, 0.7, 0.95, 0.004, 0.99, 0.02\}$. We have that $\{z, k, c, H\} = \{1, 9.9989, 0.7180, 0.271$

We then use equation (14) to substitute for z in equations (15)-(17) and then take a linear, Taylor expansion of those equations around the steady state values $\{\theta, k, c, H\}$. We end up with the following linear system

$$\theta'_{t+1} = \rho\theta'_t + v_{t+1} \tag{18}$$

$$A1c'_t + A2k'_t + A3k'_{t+1} + A4H'_t + A5\theta'_t = 0 \tag{19}$$

$$B1c'_t + E_t B2c'_{t+1} + B3k'_{t+1} + E_t B4H'_{t+1} + E_t B5\theta'_{t+1} = 0 \tag{20}$$

$$D1c'_t + D2k'_t + D3H'_t + D4\theta'_t = 0 \tag{21}$$

where the A's, B's and D's are functions of $\alpha, b, d, \gamma, \psi, \rho, \mu, k, c, \theta, H$ and thus known numbers, and $c'_t = (c_t - c)/c$ and so on. Equations (18,19,20,21) form a system of four stochastic, first order, linear difference equations.

The agents only need information on k_t and θ_t in order to decide how much to consume, work and invest in period t. k and θ are the state variables of the system (the former is endogenous while the latter is exogenous). It is then reasonable to assume that their decisions take the form

$$k'_{t+1} = h_1(k'_t, \theta'_t) \quad c'_t = h_2(k'_t, \theta'_t) \quad H'_t = h_3(k'_t, \theta'_t)$$

Our objective again is to determine the functions h_1, h_2 and h_3.

The Farmer method

Let us use equation (21) to substitute for H_t and H_{t+1} in equations (18,19,20. The resulting system can be written as

$$R1s'_{t+1} = R2s'_t + R3n'_{t+1} \tag{22}$$

where $s'_t = \{\theta'_t, k'_t, c'_t\}$ and $n'_{t+1} = \{v'_{t+1}, w'_{\theta t+1}, w'_{kt+1}, w'_{ct+1}\}$ with $w'_{it+1} = E_t w'_{it+1} - w'_{it+1}$, $i = \theta, k, c$.

If R1 is non singular then we multiply (22) through by $R1^{-1}$ to arrive at

$$s'_{t+1} = W1s'_t + W2n'_{t+1} \tag{23}$$

where

$$W1 = \begin{pmatrix} 0.9500 & 0 & 0 \\ 0.1454 & 1.0544 & 0.2351 \\ -0.0408 & 0.0083 & 0.9593 \end{pmatrix}$$

W1 can be written as

$$W1 = M^{-1}JM \tag{24}$$

where J is a diagonal matrix with elements the eigenvalues of $W1$ and M is the matrix whose columns are the eigenvectors of $W1$. In particular

$$J = \begin{pmatrix} 0.9418 & 0 & 0 \\ 0 & 0.9500 & 0 \\ 0 & 0 & 1.0718 \end{pmatrix} \quad M = \begin{pmatrix} -0.9019 & 0.8913 & 0.9973 \\ 0 & 0.0812 & 0 \\ 0.4319 & -0.4460 & 0.0739 \end{pmatrix}$$

Let us use (24) in (23), then multiply through by M and finally take expectations. The transformed system now takes the form

$$E_t x_{t+1} = J x_t \qquad x_t = M s_t \tag{25}$$

In order to have a unique stationary equilibrium we need exactly two of the roots (= # of state variables) to lie inside the unit circle (and one to lie inside). Suppose that the roots have been ranked in terms of increasing absolute value so that those inside the unit circle have been placed at the top. Subsequently, the third element of x is equal to zero (Let r_j be the $i-$th root of the characteristic polynomial. If we multiply through (25) by J^{-1} and substitute recursively inside the expectation operator then $\lim_{m\to\infty}(r_j)^{-m}E_t x_{t+m} = 0$ for $r_j > 1$. In our case $j = 3$). This element defines an equation in one unknown (c_t) that can be solved in terms of the state variables (z_t and k_t). In particular

$$c_t = 0.6207k_t + 0.1789z_t$$

Substituting this solution in (21) gives the solution for H_t

$$H_t = -0.3751k_t + 0.6528z_t$$

These solutions for c_t and H_t can be used in (19) to obtain the solution for k_{t+1}. It is

$$k_{t+1} = 0.9418k_t + 0.0897z_t$$