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A SIMPLE, TWO SECTOR, TWO FACTOR, ENDOGENOUS STOCHASTIC GROWTH MODEL

Economic environment

Two sectors: Sector 1 produces a single good that can be used for consumption and investment purposes. Sector 2 produces human capital. Production uses two inputs (physical and human capital). The subscript it denotes sector i in period t:

Preferences:

$$U(C_t) = \psi \ln C_t \tag{1}$$

Goods production:

$$Y_t = F(K_{1t}; H_{1t}; z_{1t}) = z_{1t}(u_{1t}K_t)^a (N_{1t}H_t)^{1-\alpha} \tag{2}$$

The economy's resource constraint is

$$Y_t = F(K_{1t}; H_{1t}; z_{1t}) = C_t + I_t \tag{3}$$

Physical capital depreciates at the rate of d_k per period (the same in both sectors)

The physical capital stock evolves as follows

$$K_{t+1} = (1 - d_k)K_t + I_t \quad = (1 - d_k)K_t + z_{1t}(u_{1t}K_t)^a (N_{1t}H_t)^{1-\alpha} - C_t \tag{4}$$

Human capital production

$$X_t = G(K_{2t}; H_{2t}; z_{2t}) = z_{2t}(u_{2t}K_t)^w (N_{2t}H_t)^{1-w} \tag{5}$$

The human capital stock evolves as follows

$$H_{t+1} = (1 - d_H)H_t + X_t \quad = (1 - d_H)H_t + z_{2t}(u_{2t}K_t)^w (N_{2t}H_t)^{1-w} \tag{6}$$

Note that

$$N_{1t} + N_{2t} = 1 \tag{7}$$

$$u_{1t} + u_{2t} = 1 \tag{8}$$

The technological shock in sector i evolves according to

$$\log(z_{it+1}) = (1 - \rho_i)\log(z_i) + \rho_i \log(z_{it}) + v_{it+1}; \quad v_{it+1} \ \text{iid} \tag{9}$$

Information: v_{it} becomes known in period t

The value function is

$$V(K_t; H_t; z_t) = \max\{u(C_t; N_{1t}) + bE_t V(K_{t+1}; H_{t+1}; z_t)\} \tag{10}$$

where E_t is conditional expectation taken in period t and $K_t; H_t$ and z_t are the state variables:

The objective is to maximize (10) subject to (4),(6),(7), (8), and (9) by selecting the optimal sequence $\{C_t; K_{t+1}; H_{t+1}; N_{1t}; N_{2t}; u_{1t}; u_{2t}\}_{t=0}^{1}$. Use (7), (8) in (5) and (6) to express u_{2t} and N_{2t} in terms of u_{1t} and N_{1t} respectively. Let λ_{1t};and λ_{2t} be the Langrange multipliers associated with (4) and (6). The FOC are

$$U_{Ct} = \lambda_{1t} \tag{11}$$

$$bE_t\{(1 - d_k + F_{kt+1})\lambda_{1t+1} + G_{kt+1}\lambda_{2t+1} = \lambda_{1t} \tag{12}$$

$$bE_t\{F_{Ht+1}\lambda_{1t+1} + (1 - d_H + G_{Ht+1})\lambda_{2t+1} = \lambda_{2t} \tag{13}$$

$$\lambda_{1t}F_{N1t} + \lambda_{2t}G_{N1t} = 0 \tag{14}$$

$$\Lambda_{1t}F_{u1t} + \Lambda_{2t}G_{u1t} = 0 \tag{15}$$

Equations(11)-(15) together with (4), (6) and (9) form a system of equations that fully characterizes the equilibrium of this economy. Because the economy experiences sustained -endogenous- growth, the levels of the variables K, C and H are nonstationary. We can get rid of nonstationarity by expressing variables in terms of human capital. The corresponding system is

$$g1(c_t, \lambda_{1t}) = 0 \tag{16}$$

$$E_t g2(k_{t+1}, \lambda_{1t+1}, \lambda_{2t+1}, u_{1t+1}, N_{1t+1}, z_{1t+1}, z_{2t+1}) = 0 \tag{17}$$

$$E_t g3(k_{t+1}, \lambda_{1t+1}, \lambda_{2t+1}, u_{1t+1}, N_{1t+1}, z_{1t+1}, z_{2t+1}) = 0 \tag{18}$$

$$g4(k_t, \lambda_{1t}, \lambda_{2t}, u_{1t}, N_{1t}, z_{1t}, z_{2t}) = 0 \tag{19}$$

$$g5(k_t, \lambda_{1t}, \lambda_{2t}, u_{1t}, N_{1t}, z_{1t}, z_{2t}) = 0 \tag{20}$$

$$g6(k_{t+1}, k_t, u_{1t}, N_{1t}, z_{1t}, z_{2t}) = 0 \tag{21}$$

$$g7(z_{1t+1}, z_{1t}, v_{1t+1}) = 0 \tag{22}$$

$$g8(z_{2t+1}, z_{2t}, v_{2t+1}) = 0 \tag{23}$$

where $k_t = K_t/H_t$, $c_t = C_t/H_t$ and $\lambda_{it} = \Lambda_{it}H_t$. We now proceed as in the case of the one factor growth model. We first calculate the steady state solution $\{z_1, z_2, k, c, u_1, N_1, \lambda_1, \lambda_2\}$. Let $\{\rho_1, \rho_2, \alpha, w, \psi, b, d_K, d_H\} = \{0.95, 0.95, 0.35, 0.3, 1, 0.99, 0.02, 0.01\}$. The steady state vector is then $\{z_1, z_2, k, c, u_1, N_1, \lambda_1, \lambda_2\} = \{1, 0.0123, 33.7072$ and the growth rate of the economy is $\mu = 1.0035$.

We then take a linear, Taylor expansion of equations (16)- (23)) around the steady state $\{z_1, z_2, k, c, u_1, N_1, \lambda_1, \lambda_2\}$. We end up with the following system of linear, first order, expectational difference equations

$$z'_{1t+1} = \rho_1 z'_{1t} + v_{1t+1} \tag{24}$$

$$z'_{2t+1} = \rho_2 z'_{2t} + v_{2t+1} \tag{25}$$

$$A1c'_t + A2\lambda'_t = 0 \tag{26}$$

$$E_t\{B1k'_{t+1} + B2\lambda'_{1t+1} + B3\lambda'_{2t+1} + B4u'_{1t+1} + B5N'_{1t+1} + B6z'_{1t+1} + B7z'_{2t+1}\} = 0 \tag{27}$$

$$E_t\{D1k'_{t+1} + D2\lambda'_{1t+1} + D3\lambda'_{2t+1} + D4u'_{1t+1} + D5N'_{1t+1} + D6z'_{1t+1} + D7z'_{2t+1}\} = 0 \tag{28}$$

$$L1B1k'_t + L2\lambda'_{1t} + L3\lambda'_{2t} + L4u'_{1t} + L5N'_{1t} + L6z'_{1t} + L7z'_{2t} = 0 \tag{29}$$

$$Q1B1k'_{t+1} + Q2k'_t + Q3u'_{1t} + Q4N'_{1t} + Q5z'_{1t} + Q6z'_{2t} = 0 \tag{30}$$

$$T1B1k'_{t+1} + T2k'_t + T3u'_{1t} + T4N'_{1t} + T5z'_{1t} + T6z'_{2t}\} = 0 \tag{31}$$

where the A, B, D,L, Q, T's are functions of $\theta_1, \theta_2, \rho_1, \rho_2, \alpha, w, \psi, b, d_K, d_H, z_1, z_2, k, c, u_1, N_1, \lambda_1, \lambda_2$ and thus known numbers, and $c'_t = (c_t - c)/c$ and so on.

The agents only need information on k_t and z_{it} in order to decide how much to consume, work and invest in period t. k and z_i are the (transformed) state variables of the system (the former is endogenous while the latter is exogenous). It is then reasonable to assume that their decisions take the form

$$k'_{t+1} = h_1(k'_t, z'_{1t}, z'_{2t}) \quad c'_t = h_2(k'_t, z'_{1t}, z'_{2t}) \quad N'_{it} = h_{i3}(k'_t, z'_{1t}, z'_{2t}) \quad u'_{it} = h_{i4}(k'_t, z'_{1t}, z'_{2t})$$

Our objective again is to determine the functions h_1, h_2, h_{i3} and h_{i4}.

The Farmer method

The system of equations (24-31) can be written as

$$R1 s'_{t+1} = R2 s'_t + R3 n'_{t+1} \tag{32}$$

where

$s'_t = \{z'_{1t}, z'_{2t}, k'_t, c'_t, u'_{t1}, N'_{t1}, \lambda'_{t1}, \lambda'_{t2}\}$ and

$n'_{t+1} = \{v'_{1t+1}, v'_{2t+1}, E_t k'_{t+1} - k'_{t+1}, E_t c'_{t+1} - c'_{t+1}, E_t u'_{1t+1} - u'_{1t+1}, E_t N'_{1t+1} - N'_{1t+1}, E_t \lambda'_{1t+1} - \lambda'_{1t+1}, E_t \lambda'_{2t+1} - \lambda'_{2t+1}\}$

and the elements of R1; R1; R3 are taken from the corresponding variables in equations (24-31).

If R1 is non singular then we multiply (32) through by $R1^{-1}$ to arrive at

$$s'_{t+1} = W1 s'_t + W2 n'_{t+1} \tag{33}$$

W1 can be written as

$$W1 = M^{-1} J M \tag{34}$$

where J is a diagonal matrix with elements the eigenvalues of W1 and M is the matrix whose columns are the eigenvectors of W1. In particular

Let us use (34) in (33), then multiply through by M and finally take expectations. The transformed system now takes the form

$$E_t x_{t+1} = J x_t \quad \text{where } x_t = M s'_t \tag{35}$$

where the first element (row) of x, say, x1, is simply the product of the first row of M times the vector column s'_t and so on. Note that the elements (rows) of x_t that correspond to the eigenvalues that lie outside the unit circle are equal to zero because -as can be easily seen by multiplying in (??) by J^{-1} and recursively substituting for the expectation in the LHS- the corresponding left hand side is equal to zero. But each element of x defines a linear relationship between the elements of s'_t. Subsequently, using these linear relationships we can solve for the variables $c'_t, k'_{t+1}, u'_{t1}, N'_{t1}, \lambda'_{t1}, \lambda'_{t2}$ as a function of z'_{1t}, z'_{2t}, k'_t. The solutions are

$$c'_t = 0.6228 k'_t + 0.2464 z'_{1t} - 0.0972 z'_{2t}$$

$$k'_{t+1} = 0.5674 k'_t + 0.9685 z'_{1t} - 0.1702 z'_{2t}$$

$$u'_{1t} = -0.1271 k'_t + 0.1953 z'_{1t} - 0.2431 z'_{2t}$$

$$N'_{1t} = -0.1467 k'_t + 0.2254 z'_{1t} - 0.2805 z'_{2t}$$

$$\lambda'_{1t} = -0.6228 k'_t - 0.2464 z'_{1t} + 0.0972 z'_{2t}$$

$$\lambda'_{2t} = -0.1200 k'_t + 0.0579 z'_{1t} - 0.0369 z'_{2t}$$

Moreover

$$y'_t = 0.2102 k'_t + 1.2149 z'_{1t} - 0.2674 z'_{2t}$$

$$i'_t = -0.4126 k'_t + 0.9685 z'_{1t} - 0.1702 z'_{2t}$$