Doing Economics with the Computer

Special Matlab session: Solving linear rational expectations models

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Topic of today

In macroeconomics we frequently encounter so-called **linear rational expectations (RE) models**. There exists a number of algorithms by means of which these models can be solved numerically. The one we are going to look at is particularly straightforward and intuitive.

It is based on Roger E. A. Farmer, *The macroeconomics of self-fulfilling prophecies* [an introductory textbook into modern macroeconomics for advanced undergrads or beginning graduate students].

The basic framework

Consider the following linear stochastic first-order vector system

$$Y_t = A \cdot Y_{t+1} + B \cdot X_{t+1} \tag{1}$$

where (i) Y_t is a vector of **state variables**. It is devided into **free** and **predetermined** variables. A predetermined variable is defined as a variable that is associated with an initial condition. (ii) X_t is a vector of two kinds of variables. First, a set of **fundamental disturbances**, V_t, that are i.i.d. through time and for which we have $E\left[V_t\right] = \bar{V}$. Second, deviations of the vector of state variables from their one-step-ahead expected values (**expectational errors**), $w_t^{y^i} = E_{t-1}\left[y_t^i\right] - y_t^i$ for $i = 1, 2, ..., n$ where n is the number of state variables. (iii) A and B are **matrices of coefficients**.

An illustrative example: The basic RBC model with exogenous labor supply

On example of a RE model is the basic RBC model:

The yeoman farmer maximizes expected life-time utility

$$E_t \left\{ \sum_{t=1}^{\infty} \beta^{t-1} \log\left(c_t\right) \right\}$$

subject to

$$\begin{aligned} s_t k_t^{\alpha} &= c_t + i_t \\ k_{t+1} &= (1-\delta)k_t + i_t \\ s_t &= s_{t-1}^{\rho} v_t \end{aligned}$$

with initial values $k_0 = \bar{k}_0$ and $s_0 = \bar{s}_0$.

c_t	consumption
i_t	(gross) investment
k_t	physical capital stock
s_t	total factor productivity
v_t	technology shock (white noise)
$\alpha \in (0,1)$	share of physical capital in production
$\beta \in [0,1)$	discount factor
$\delta \in (0,1)$	rate of depreciation
$\rho \in [0,1)$	autoregressive coefficient

Notation:

Y_t; y_t	vector of variables; variable
A; a_{ij}	matrix of coefficients; element (ij) in matrix A

Solving the dynamic optimization problem

The solution to the dynamic optimization problem (or the equilibrium to this economy) is given by the **FOCs** to the problem

$$\max_{c_t} E_t \left\{ \sum_{t=1}^{\infty} \beta^{t-1} \log(c_t) \right\}$$

subject to

$$s_t k_t^{\alpha} = c_t + k_{t+1} - (1-\delta)k_t$$

plus the **(stochastic) process** which describes the evolution of the exogenous variable s_t

$$s_t = s_{t-1}^{\rho} v_t$$

The FOCs is given by (for a derivation see a separate manuscript on dynamic optimization)

$$\frac{1}{c_t} = \beta E_t \left\{ \frac{1}{c_{t+1}} \left(1 - \delta + \alpha s_{t+1} k_{t+1}^{\alpha-1}\right) \right\}$$

$$k_{t+1} = (1 - \delta)\, k_t + s_t k_t^{\alpha} - c_t$$

$$s_{t+1} = s_t^{\rho} v_{t+1}$$

Note that the resulting system of equations is **non-linear**. In what follows, it will be linearized around the steady state.

But first we have to derive the steady state equilibrium.

Steady-state equilibrium

A steady-state equilibrium for this economy is one in which the technology shocks is assumed to be silent (constant) so that there is no uncertainty. We set $v_t = 1$ for all t. The values of consumption, capital, and the total factor productivity are constant, $c_t = \bar{c}$, $k_t = \bar{k}$, and $s_t = \bar{s}$ for all t. The solutions to $\bar{c}$, $\bar{k}$, and $\bar{s}$ can be expressed as follows:

$$\frac{\bar{k}}{\bar{y}} = \frac{\alpha\beta}{1-\beta(1-\delta)}$$

$$\frac{\bar{i}}{\bar{y}} = \delta\frac{\bar{k}}{\bar{y}}$$

$$\frac{\bar{c}}{\bar{y}} = 1 - \frac{\bar{i}}{\bar{y}}$$

$$\bar{y} = \left(\frac{\bar{k}}{\bar{y}}\right)^{\frac{\alpha}{1-\alpha}}$$

First-order Taylor approximation

Next, we derive a **first-order Taylor approximation** of the FOCs and the stochastic process **around the steady state**.

The linearized FOCs are given by (for a derivation see a separate manuscript on linearization)

$$-\hat{c}_t = E_t \left\{ a_1 \hat{k}_{t+1} + a_2 \hat{s}_{t+1} - \hat{c}_{t+1} \right\}$$

$$\hat{k}_{t+1} = b_1 \hat{k}_t + b_2 \hat{s}_t + b_3 \hat{c}_t$$

The linearized stochastic process is given by

$$\hat{s}_{t+1} = \rho \hat{s}_t + \hat{v}_{t+1}$$

where $\hat{c}_t$ e.g. denotes percentage deviations from the steady-state value of c_t:

$$\hat{c}_t = \frac{c_t - \bar{c}}{\bar{c}}$$

and the parameters a_1, a_2, b_1, b_2, and b_3 are functions of the structural parameters (α, β, etc.) and the steady state values of k_t and s_t.

$$a_1 = \beta\alpha\left(\alpha - 1\right)\bar{s}\bar{k}^{\alpha-1}$$

$$a_2 = \beta\alpha\bar{s}\bar{k}^{\alpha-1}$$

$$b_1 = 1 - \delta + \alpha\bar{s}\bar{k}^{\alpha-1}$$

$$b_2 = \alpha\bar{s}\bar{k}^{\alpha-1}$$

$$b_3 = -\frac{\bar{c}}{\bar{k}}$$

Casting the RBC model in framework (1)

First, we rearrange the linearized FOCs and the linearized stochastic process in order to cast them into the system

$$F \cdot Y_t = G \cdot Y_{t+1} + H \cdot X_{t+1}$$

where $Y = \begin{bmatrix} \hat{c} & \hat{k} & \hat{s} \end{bmatrix}'$ and $X = \begin{bmatrix} \hat{v} & w^{\hat{c}} & w^{\hat{k}} & w^{\hat{s}} \end{bmatrix}'$.

Second, we transform the system $FY_t = GY_{t+1} + HX_{t+1}$ into framework (1)

$$\begin{aligned} FY_t &= GY_{t+1} + HX_{t+1} \\ &\rightsquigarrow \quad Y_t = AY_{t+1} + BX_{t+1} \end{aligned}$$

where $A \equiv F^{-1}G$ and $B \equiv F^{-1}H$.

To this aim, the Euler equation has to be rearranged as follows

$$\begin{aligned}
-c_t &= -E_t\left\{c_{t+1}\right\} + a_1 E_t\left\{k_{t+1}\right\} + a_2 E_t\left\{s_{t+1}\right\} \\
-c_t &= -E_t\left\{c_{t+1}\right\} + c_{t+1} - c_{t+1} + a_1 E_t\left\{k_{t+1}\right\} + a_1 k_{t+1} - a_1 k_{t+1} \\
&\quad + a_2 E_t\left\{s_{t+1}\right\} + a_2 s_{t+1} - a_2 s_{t+1} \\
-c_t &= -c_{t+1} + a_1 k_{t+1} + a_2 s_{t+1} - w_{t+1}^{c} + a_1 w_{t+1}^{k} + a_2 w_{t+1}^{s}
\end{aligned}$$

The restructured FOCs are given by

$$\begin{aligned}
-\hat{c}_t &= -\hat{c}_{t+1} + a_1 \hat{k}_{t+1} + a_2 \hat{s}_{t+1} - w_{t+1}^{\hat{c}} + a_1 w_{t+1}^{\hat{k}} + a_2 w_{t+1}^{\hat{s}} \\
b_3 \hat{c}_t + b_1 \hat{k}_t + b_2 \hat{s}_t &= \hat{k}_{t+1} \\
\rho \hat{s}_t &= \hat{s}_{t+1} - \hat{v}_{t+1}
\end{aligned}$$

In Farmer's terminology, $\hat{c}_t$ is a free variable and $\hat{k}_t$ and $\hat{s}_t$ are predetermined state variables.

Thus, we have

$$F = \begin{bmatrix} -1 & 0 & 0 \\ b_3 & b_1 & b_2 \\ 0 & 0 & \rho \end{bmatrix}; G = \begin{bmatrix} -1 & a_1 & a_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$H = \begin{bmatrix} 0 & -1 & a_1 & a_2 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

Solution method: Key idea

To the system

$$\begin{bmatrix} \hat{c}_t \\ \hat{k}_t \\ \hat{s}_t \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \hat{c}_{t+1} \\ \hat{k}_{t+1} \\ \hat{s}_{t+1} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \end{bmatrix} \begin{bmatrix} \hat{v}_t \\ w^{\hat{c}_t} \\ w^{\hat{k}_t} \\ w^{\hat{s}_t} \end{bmatrix}$$

we look for the recursive representation solution of the form

$$\hat{c}_t = \Pi \cdot \tilde{Y}_t$$

$$\tilde{Y}_t = \Psi \cdot \tilde{Y}_{t-1} + \Theta \cdot \hat{v}_t \tag{2}$$

where $\tilde{Y} = \begin{bmatrix} \hat{k} & \hat{s} \end{bmatrix}'$. That is, the solution to system (1) is described by the matrices Π, Ψ, and Θ (for non-Greeks: "Pi", "Psi", and "Theta").

The model is regular if

$$n_s + n_1 = n$$

where n_s is the number of eigenvalues inside the unit circle of **matrix** A, n_1 is the number of **predetermined variables**, and n is the number of **state variables**.

In the following we assume that $\alpha = 0.4$, $\beta = 0.99$, $\delta = 0.025$, and $\rho = 0.95$. It follows that A has exactly **one** eigenvalue inside the unit circle (we are going to check this). Moreover, as we have seen above, there is exactly **one** free variable, namely $\hat{c}_t$. We conclude that the model for the baseline parametrization is regular.

Framework (1) in canonical variables form

In the case at hand, the **right eigenvector decomposition** of A (a 3×3 matrix) is given by

$$A = Q \Lambda Q^{-1}$$

where

$$\Lambda \equiv \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

and

$$Q = \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{bmatrix} ; Q^{-1} = \begin{bmatrix} q^{11} & q^{12} & q^{13} \\ q^{21} & q^{22} & q^{23} \\ q^{31} & q^{32} & q^{33} \end{bmatrix}$$

With the diagonalized matrix A at our disposal we can transform framework (1) as follows

$$\begin{aligned} Y_t &= AY_{t+1} + BX_{t+1} \\ Q^{-1}Y_t &= Q^{-1}AY_{t+1} + Q^{-1}BX_{t+1} \\ Q^{-1}Y_t &= Q^{-1}Q\Lambda Q^{-1}Y_{t+1} + Q^{-1}BX_{t+1} \\ Z_t &= \Lambda Z_{t+1} + \Phi_{t+1} \qquad (3) \end{aligned}$$

where $Z_t \equiv Q^{-1}Y_t$ and $\Phi_t \equiv Q^{-1}BX_t$.

Solution

Let us have a closer look at framework (3) (still, in the case at hand):

$$\begin{bmatrix} z_t^1 \\ z_t^2 \\ z_t^3 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} z_{t+1}^1 \\ z_{t+1}^2 \\ z_{t+1}^3 \end{bmatrix} + \begin{bmatrix} \phi_{t+1}^1 \\ \phi_{t+1}^2 \\ \phi_{t+1}^3 \end{bmatrix}$$

Since Λ is a diagonal matrix, system (3) implies i independent linear difference equations of the form

$$z_t^i = \lambda_i z_{t+1}^i + \phi_{t+1}^i \tag{4}$$

for $i = 1, 2, 3$. Taking conditional expectations on both sides of (4) yields

$$z_t^i = \lambda_i E_t \left\{ z_{t+1}^i \right\} \tag{5}$$

(Note that $E_t \left\{ \phi_{t+1}^i \right\} = 0$.)

Suppose $|\lambda_i| < 1$. Then, we can iterate (5) into the *future* by means of recursive substitution

$$
\begin{aligned}
z_t^i &= \lambda_i E_t \left\{\lambda_i E_{t+1} \left\{z_{t+2}^i\right\}\right\} = \lambda_i^2 E_t \left\{z_{t+2}^i\right\} \\
z_t^i &= \lambda_i^2 E_t \left\{\lambda_i E_{t+2} \left\{z_{t+3}^i\right\}\right\} = \lambda_i^3 E_t \left\{z_{t+3}^i\right\} \\
z_t^i &= \ldots \\
z_t^i &= \lambda_i^T E_t \left\{z_{t+T}^i\right\}
\end{aligned}
$$

Since

$$\lim_{T\to\infty} \lambda_i^T = 0$$

it follows that

$$z_t^i = 0$$

Let us rearrange Q^{-1} such that the first row is associated with the stable eigenvector of A. Recall that

$$\begin{bmatrix} z_t^1 \\ z_t^2 \\ z_t^3 \end{bmatrix} = \begin{bmatrix} q^{11} & q^{12} & q^{13} \\ q^{21} & q^{22} & q^{23} \\ q^{31} & q^{32} & q^{33} \end{bmatrix} \begin{bmatrix} \hat{c}_t \\ \hat{k}_t \\ \hat{s}_t \end{bmatrix}$$

From $z_t^1 = 0$, we get the solution for $\hat{c}_t$

$$\begin{aligned} 0 &= q^{11}\hat{c}_t + q^{12}\hat{k}_t + q^{13}\hat{s}_t \\ \rightsquigarrow & \quad \hat{c}_t = -\frac{q^{12}}{q^{11}}\hat{k}_t - \frac{q^{13}}{q^{11}}\hat{s}_t \end{aligned}$$

Mind the elegancy of the argumentation!

Thus, in the case at hand matrix Π is given by

$$
\begin{aligned}
\hat{c}_t &= \begin{bmatrix} -q^{12}/q^{11} & -q^{13}/q^{11} \end{bmatrix} \begin{bmatrix} \hat{k}_t \\ \hat{s}_t \end{bmatrix} \\
&= \begin{bmatrix} \pi_{11} & \pi_{12} \end{bmatrix} \begin{bmatrix} \hat{k}_t \\ \hat{s}_t \end{bmatrix} \\
&= \Pi \cdot \tilde{Y}_t \qquad (6)
\end{aligned}
$$

In order to get matrices Ψ and Θ, we substitute the solution for $\hat{c}_t$, given by (6), into the solution for $\hat{k}_{t+1}$ and rearrange things a little bit

$$
\begin{aligned}
\hat{k}_{t+1} &= b_3\hat{c}_t + b_1\hat{k}_t + b_2\hat{s}_t \\
&= b_3\left(\pi_1\hat{k}_t + \pi_2\hat{s}_t\right) + b_1\hat{k}_t + b_2\hat{s}_t \\
&= \left(b_1 + b_3\pi_1\right)\hat{k}_t + \left(b_2 + b_3\pi_2\right)\hat{s}_t
\end{aligned}
$$

Moreover, we recall that $\hat{s}_t = \rho\hat{s}_{t-1} + \hat{v}_t$.

Thus, in the case at hand system (2) is given

$$
\begin{aligned}
\begin{bmatrix} \hat{k}_t \\ \hat{s}_t \end{bmatrix} &= \begin{bmatrix} b_1 + b_3\pi_1 & b_2 + b_3\pi_2 \\ 0 & \rho \end{bmatrix} \begin{bmatrix} \hat{k}_{t-1} \\ \hat{s}_{t-1} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \hat{v}_t \\
\begin{bmatrix} \hat{k}_t \\ \hat{s}_t \end{bmatrix} &= \begin{bmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{bmatrix} \begin{bmatrix} \hat{k}_{t-1} \\ \hat{s}_{t-1} \end{bmatrix} + \begin{bmatrix} \theta_{11} \\ \theta_{21} \end{bmatrix} \hat{v}_t \\
\tilde{Y}_t &= \Psi\tilde{Y}_{t-1} + \Theta\hat{v}_t
\end{aligned}
$$

We are done!

Today's task

- Find the solution to the basic RBC with the baseline parametrization. (Define the algorithm parameters; Define the structural parameters of the model; Define the steady state values and the parameters a_i and b_j; Define the matrices F, G, and H; From this derive matrix A; Come up with the eigenvector decomposition of A, i.e., with matrices Q and Λ; Check the regularity condition; Rearrange Q^{-1} such that the first row is associated with the stable eigenvector of A; Compute Π; Compute Ψ and Θ.)
- Compute and plot the IRF for c_t and k_t.
- Optional: Compute and plot the IRF for y_t and i_t.
- Repeat the exercise for the case where the technology shock is very short lived ($\rho = 0$). Where depreciation is complete ($\delta = 1$).