The one-good, one-shock RBC model (2)

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1 Introduction

> "We think about models not because they are realistic, but because thinking about models is a good warm-up exercise for thinking about the world we live in. The goal, always, is to understand our own world."
>
> Steven T. Landsburg (1993), *The armchair economist*

In this summary, we develop the working horse in modern business cycle theory: the market-clearing or real business cycle (RBC) model. We start with the labor-leisure choice in a static setting. We study an individual that maximizes his or her utility subject to a resource constraint. Next, we focus on the individual consumption choice problem within a dynamic setting. We study an individual that maximizes lifetime utility subject to an asset accumulation condition. Third, we consider a preliminary version of the business cycle model, where the economy as a whole has no possibility to save or dissave and must satisfy the economy's resource constraint in each an every period. Despite these limitations the model gives many insights of how a dynamic stochastic general equilibrium model works. Finally, we introduce capital and investment and, thus, the possibility of a closed economy to save and dissave. We compare the predictions of the full-fledged RBC model with the business cycle facts outlined in Summary 1. The summary draws on Barro [1] and King and Rebelo [3], among others.

2 Labor/leisure choice

Suppose, households are isolated from each other and therefore behave like Robinson Crusoe's. There are no markets on which people can trade, and each household uses its own labor to produce goods by means of a production function. Because we treat goods as nonstorable, each household consumes what it produces.

2.1 The Robinson-Crusoe economy

FIGURE 1: COMBINING INDIFFERENCE CURVE AND RESOURCE CONSTRAINT

We begin by considering Robinson Crusoe's labor/leisure choice. Suppose Robinson has continuous, convex, and strongly monotone preferences $\succeq$ defined over a (single) consumption good, c, and his consumption of leisure, l. He has an endowment of $\bar{L}$ units of time (which we typically normalize to 1) and no endowment of the consumption good. Let $u(c, l)$ be a utility function representing $\succeq$. (Note that the assumptions on Robinson's preferences imply that $u(c, l)$ is strictly increasing, strictly concave, and continuously differentiable.)

Robinson uses his labor effort as an input to production (he has no capital or any other inputs). The quantity of Robinson's output per period, y, is a function of the quantity of labor input, n

$$y = f(n)$$

We assume $f(\cdot)$ to be strictly increasing and strictly concave. Since Robinson has no opportunity to exchange commodities, his only option is to consume all the goods that he produces in each period

$$c = y = f(n)$$

Robinson's problem is given as follows:

$$\max_{c,l \geq 0} u(c,l)$$

subject to

$$\begin{aligned} c &= f(n) \\ n &= 1 - l \end{aligned}$$

Robinson selects from the attainable combinations of work and consumption (given by the production function) the one that gives him the most utility (i.e. the highest indifference curve) (compare Figure 1). In the optimum, the slope of the production function equals the slope of the indifference curve (check this in the graph):

$$f'(n) = \underbrace{\frac{u_2(c,l)}{u_1(c,l)}}_{MRS}$$

2.2 Digression: The one-consumer, one-producer economy

Suppose that there are two price-taking economic agents, a single consumer and a single firm, and two goods, the labor (or leisure) of the consumer and a (non-storable) consumption good produced by the firm. *The firm* uses labor to produce the consumption good according to the increasing and strictly concave production function $y = f(n)$, where n denotes work effort. Thus, to produce output, the firm must hire the consumer, effectively purchasing some of the consumer's leisure from him. We assume that the firm seeks to maximize its profits. Letting p be the price of its output (which is usually normalized to one) and w be the price of labor, the firm solves

$$\max_{n \geq 0} p \cdot f(n) - wn \tag{1}$$

Given prices (p, w), the firm's optimal labor demand is $n(p, w)$, its output is $y(p, w)$, and its profits are $\pi(p, w)$. Furthermore, we assume that the consumer is the sole owner of the firm and receives the profits earned by the firm.

The consumer has continuous, convex, and strongly monotone preferences $\succeq$ defined over the consumption good, c, and his consumption of leisure, l. He has an endowment of $\bar{L}$ units of leisure (which we normalize to 1) and no endowment of the consumption good. Letting $u(c, l)$ be a utility function representing $\succeq$, the consumer's problem given prices (p, w) is

$$\max_{c,l \geq 0} u(c, l)$$

subject to

$$\begin{aligned} pc &\leq wn + \pi(p, w) \\ n &= 1 - l \end{aligned}$$

Since the utility function is strictly increasing, the first constraint holds with equality.

A *Walrasian equilibrium* in this economy involves an allocation (c^*, l^*) and a price vector (p^*, w^*) at which

- The consumer's problem is solved
- The firm's problem is solved
- The good and labor markets clear, that is

$$c(p^*, w^*) = y(p^*, w^*)$$

  and

$$n(p^*, w^*) = 1 - l(p^*, w^*)$$

The consumer selects from the attainable combinations of work and consumption (given by the production function) the one that gives him the most utility (i.e. the highest indifference curve). This equilibrium (or efficiency) condition is given by

$$f'(n) = \underbrace{\frac{u_2(c, l)}{u_1(c, l)}}_{MRS}$$

Note that p has been normalized to 1. Then the above optimality condition requires that the slope of the production function (the marginal product of labor) equals the marginal rate of substitution (the slope of the indifference curve).

Given the convexity of preferences and the strict convexity of the aggregate production set assumed here, there is a *unique* Walrasian equilibrium allocation (compare Figure 1). Since the well-being of the single consumer is maximized subject to feasibility constraints the Walrasian equilibrium allocation is *Pareto optimal*. That is, the Walrasian equilibrium allocation is the same allocation that would be obtained if a planner ran the economy in a manner that maximized the consumer's well-being.

2.3 Comparative statics

In this section we want to understand how people alter their work effort and consumption when there are changes in the opportunities for production. We begin by noting that an extra hour on the production function generates additional output (and, hence, consumption) of

$$\lim_{\Delta n \to 0} \frac{\Delta y}{\Delta n} = \frac{\partial y}{\partial n} = f'(n)$$

units. Thus, $f'(n)$ measures the change in production resulting from a marginal change in the amount of labor supplied. Because additional time at work necessarily means less leisure time, $f'(n)$ also gives the price, in terms of leisure forgone, of increasing consumption. In other words, $f'(n)$ is the price of leisure in terms of output which has to be given up when Robinson wishes to enjoy one unit more of leisure.

Let us, first, consider a parallel upward shift of the production function. A parallel upward shift of the production function simply means that Robinson can produce and consume more goods for the same amount of work effort while the marginal product of labor remains unchanged, i.e. the relative ease or cost with which Robinson can obtain consumption good or leisure are not altered. That is, *wealth* - and only wealth - increases. Robinson responds to an increase in wealth by raising the demand for everything that provides utility, i.e. c and l. As a corollary labor supply, n, decreases.

Second, let us consider a rotation of the production function through the original equilibrium point, that is, a *'pure' substitution effect*. (We put pure in quotation-marks since its derivation is a bit awkward - see Barro [1], chapter 2. It's enough when we catch the intuition.) Such an increase of the marginal product of labor represents a change in the relative price of consumption. Since the price of leisure increases relative to the price of consumption people shift away from leisure toward consumption, i.e. they consume and work more and enjoy less leisure.

Finally, consider a proportional upward shift of the production function. A proportional upward shift induces both wealth and substitution effects. Regarding consumption both effects show in the same direction. We conclude that consumption (and output) increases. Regarding leisure time (or work effort) the two effects show in opposite directions; hence, the overall effect on labor supply is ambiguous.

3 Consumption/saving decision

In this section, we study the consumption/saving decision of an individual which has access to a bond market and can, thus, freely borrow and lend. We start with a setting in which we assume that the consumer lives for two time periods. Next, we assume that the the consumer lives for T time periods, where T can be any positive integer. Finally, we assume that the consumer lives forever.[1] In all three cases, decisions are based on perfect foresight of the future.[2]

3.1 The two period case with exogenous income

3.1.1 Optimal consumption choice over time

Consider the optimal consumption choice over time for an individual who can freely borrow or lend and who lives for two periods. Its *lifetime utility* is given by

$$U(c_1, c_2) = u(c_1) + \beta u(c_2) \tag{2}$$

where c_1 is the amount of consumption in period 1, c_2 is the amount of consumption in period 2 and $u(\cdot)$ is called *momentary utility*. We assume that the momentary utility function is strictly increasing in its argument and strictly concave: $u'(\cdot) > 0$ and $u''(\cdot) < 0$. β denotes the discount rate; it is defined as $\beta \equiv 1/(1+\rho)$, where ρ denotes the *rate of time preference*.

The momentary utility function is often specified to[3]

$$u(c_t) = \begin{cases} \frac{1}{1-\sigma}\left(c_t^{1-\sigma} - 1\right) & \text{if } \sigma > 0,\, \sigma \neq 1 \\ \ln(c_t) & \text{if } \sigma = 1 \end{cases}$$

For this specification the elasticity of intertemporal substitution between consumption at any two points in time is constant.[4] We refer to this class of util-

[1]This might sound weird to you. But, let's imaging a family dynasty that lasts forever in so far as parents are connected to their children through a pattern of intergenerational transfers so that the household plans with an infinite horizon. Then the assumption makes sense.

[2]In later chapters we will abandon the assumption of perfect foresight and introduce uncertainty about future income. As we will see this does not change the basic insights of this section.

[3]In Summary 3, we will learn more about why this specific form is convenient.

[4]The elasticity of intertemporal substitution is given by the *absolute value* of

$$\eta_{t,s} = \frac{d\left[\frac{c_t}{c_s}\right]}{d\left[\frac{u'(c_t)}{u'(c_s)}\right]} \cdot \frac{\left[\frac{u'(c_t)}{u'(c_s)}\right]}{\left[\frac{c_t}{c_s}\right]}$$

Let $z \equiv \left[\frac{c_t}{c_s}\right]^{-\sigma}$ and, thus, $\left[\frac{c_t}{c_s}\right] = z^{-\frac{1}{\sigma}}$. Further note that $u'(c_t) = c_t^{-\sigma}$ and $\left[\frac{u'(c_t)}{u'(c_s)}\right] = \left[\frac{c_t}{c_s}\right]^{-\sigma} (\equiv z)$. Hence, $\eta_{t,s} = \frac{d\left[z^{-\frac{1}{\sigma}}\right]}{dz} \cdot \frac{z}{z^{-\frac{1}{\sigma}}} = -\frac{1}{\sigma} z^{-\frac{1}{\sigma}-1} \cdot z \cdot z^{\frac{1}{\sigma}} = -\frac{1}{\sigma}$. The elasticity of substitution between consumption in two periods is given by $\frac{1}{\sigma}$.

ity functions as the *isoelastic* or *constant relative risk aversion* (CRRA) class (since the elasticity of intertemporal substitution is defined as the inverse of the coefficient of relative risk aversion). (For an interpretation of the elasticity of intertemporal substitution compare Problem Set 3.)

In general, the household's period-by-period budget constraint is given by

$$b_{t+1} + c_t \leq y_t + (1 + r_t)b_t$$

where c_t is period t consumption, y_t is the exogenous income of period t, b_t is the stock of assets (or liabilities) of period t, and r_t is the real rate of return. On the right hand side, we have the *source of funds* (income plus receipts of principal plus interest for the bonds), on the left hand side we have the *use of funds* (consumption plus the stock of bonds to be carried over to the next period).

In the two period model, we typically assume that our individual starts with no wealth and does not want to leave any uncollected claims or unpaid debts, respectively. Thus, in period 1 we have

$$b_2 + c_1 \leq y_1$$

and in period 2 we have

$$c_2 \leq y_2 + (1 + r_2)b_2$$

Since r_2 is the only interest rate that plays a role in our model we set $r_2 = r$.

The two period-by-period budget constraints collapse to the following *present value lifetime budget constraint* (PVBC)

$$\begin{aligned} c_2 &\leq y_2 + (1 + r)(y_1 - c_1) \\ \rightsquigarrow c_1 + \frac{1}{1+r}c_2 &\leq y_1 + \frac{1}{1+r}y_2 \end{aligned} \tag{3}$$

The individuals dynamic choice problem is to maximize (2) subject to (3). This problem is very similar to the standard utility maximization problem in the two-goods case.

Suppose that $u(\cdot)$ is continuously differentiable. The indifference curves satisfy

$$u(c_1) + \beta u(c_2) = \bar{U}$$

Their slopes, given by

$$\begin{aligned} u'(c_1)\,dc_1 + \beta u'(c_2)\,dc_2 &= d\bar{U} = 0 \\ \rightarrow \quad \frac{dc_2}{dc_1} &= -\frac{u'(c_1)}{\beta u'(c_2)} \end{aligned}$$

show the individual's attitude toward different combinations of the consumption levels, i.e. reveal the individual's willingness to exchange consumption in period 1 for consumption in period 2.

Further, suppose that $u(\cdot)$ is strictly increasing in its argument. Then the PVBC holds with equality and can be rearranged as follows

$$c_2 = (1+r)\, y_1 + y_2 - (1+r)\, c_1$$

Since the individual can always consume his or her income, the point (y_1, y_2) is on the budget line (compare Figure 2). But there are other (c_1, c_2) combinations that are feasible too and that probably provide more utility than (y_1, y_2). However, these other (c_1, c_2) combinations involve borrowing (i.e., $c_1 > y_1$) or lending (i.e., $c_1 < y_1$). The slope of the budget constraint is given by $\frac{\partial c_2}{\partial c_1} = -(1+r)$: giving up one unit of first-period consumption allows the individual to increase second-period consumption by $1+r$.

As in the classical utility maximization problem the optimal consumption point is given by

$$\begin{aligned} (1+r) &= \frac{u'(c_1)}{\beta u'(c_2)} \\ &\rightsquigarrow u'(c_1) = (1+r)\,\beta u'(c_2) \end{aligned}$$

i.e., the condition that $(-1)\cdot MRS$ equals the slope of the budget line.

In the case of a dynamic choice problem, this efficiency condition is called an *intertemporal Euler equation*. This Euler equation, which will recur in many guises, has a simple interpretation: At a utility maximum, the individual cannot gain from feasible shifts of consumption between periods. A one-unit reduction in first-period consumption lowers U by $u'(c_1)$ (marginal utility cost). The consumption unit thus saved can be converted (by lending it) into $1+r$ units of second-period consumption that raise U by $(1+r)\,\beta u'(c_2)$ (sure marginal utility gain). An intertemporally optimal consumption plan, thus, equates the cost of forgone consumption today and the benefits of increased future consumption. An important special case is the one in which $\rho = r$ and thus $\beta = 1/(1+r)$. In this case the Euler equation becomes $u'(c_1) = u'(c_2)$, which implies that the consumer desires a flat lifetime consumption path, $c_1 = c_2$.

FIGURE 2: INTEREST RATE AND SAVING IN THE TWO-PERIOD CASE (TAKEN FROM ROMER [8], FIG 7.2)

3.1.2 Comparative statics

When r rises as in Figure 2, the budget constraint continues to go through (y_1, y_2) but becomes steeper; because the individual can obtain more units of consumption next period for each unit of consumption forgone, the cost of current consumption relative to that of next period's consumption (or intertemporal relative price of consumption) increases. In Figure 2, savings is initially zero. In this case the increase in r has no wealth effect (i.e., the individual's initial consumption bundle continues to be on the budget constraint); rather, there is a pure intertemporal substitution effect. Thus, c_1 falls, and so saving (in period 1) rises. A rise in the exogenous income leads to a parallel outward shift of the budget line. The new budget line allows the household to reach a higher indifference curve (*pure wealth effect*); both, c_1 and c_2, increase.

3.2 Digression: The finite time horizon case with exogenous income

Consider the optimal consumption choice over time for an individual who lives for T periods and who can freely borrow or lend. Its (time-separable) life-time utility is given by

$$U = \sum_{t=0}^{T} \beta^t u(c_t)$$

The individual's asset, a_t, evolves through time according to the following period-by-period (or accumulation) constraint

$$\frac{1}{1+r_t} a_{t+1} + c_t = y_t + a_t$$

where r_t denotes the period t interest rate and y_t denotes the individual's period t income (still exogenous). We use here a slightly different notation compared to the intuitive example. This time the source of funds is given by the exogenous income and the stock of assets or liabilities, respectively. The use of funds is given by consumption and the stock of bonds carried over to the next period times their price in terms of the consumption good. Do not bother, this is just an alternative way of stating the same.

In addition to the accumulation constraint we require that the terminal asset, a_{T+1}, must not be negative

$$a_{T+1} \geq 0$$

We also assume that a_0 is given.

It is convenient to rewrite the accumulation constraint in terms of the following difference equation

$$a_{t+1} = (1+r_t)(a_t + y_t - c_t)$$

which can be solved as follows. For period 1, we get

$$a_1 = (1+r_0)(a_0 + y_0 - c_0)$$

For period 2 we get

$$\begin{aligned} a_2 &= (1+r_1)(a_1 + y_1 - c_1) \\ &= (1+r_1)[(1+r_0)(a_0 + y_0 - c_0) + y_1 - c_1] \\ &= [(1+r_1)(1+r_0)](a_0 + y_0 - c_0) \\ &\quad + (1+r_1)(y_1 - c_1) \end{aligned}$$

Finally, going out all the way to the last period we find that

$$\begin{aligned} a_{T+1} &= [(1+r_T)(1+r_{T-1})\cdots(1+r_0)](a_0+y_0-c_0) \\ &+[(1+r_T)(1+r_{T-1})\cdots(1+r_1)](y_1-c_1) \\ &+[(1+r_T)(1+r_{T-1})\cdots(1+r_2)](y_2-c_2) \\ &+... \\ &+[(1+r_T)(1+r_{T-1})](y_{T-1}-c_{T-1}) \\ &+[(1+r_T)](y_T-c_T) \end{aligned} \tag{4}$$

Next, let us define the *market discount factor* (or simply: price) for date t consumption on date 0

$$p_t \equiv \frac{1}{(1+r_0)(1+r_1)\cdots(1+r_{t-1})}$$

With this discount factor at our disposal we can rewrite (4) as

$$\begin{aligned} p_T a_{T+1} &= (a_0+y_0-c_0) \\ &+p_1(y_1-c_1) \\ &+p_2(y_2-c_2) \\ &+... \\ &+p_T(y_T-c_T) \\ p_T a_{T+1} &= a_0+y_0-c_0+\sum\nolimits_{t=1}^{T} p_t(y_t-c_t) \end{aligned}$$

From the *terminal asset condition* we get

$$\begin{aligned} \underbrace{\frac{1}{p_T}\left[a_0+\sum\nolimits_{t=0}^{T} p_t(y_t-c_t)\right]}_{a_{T+1}} &\geq 0 \\ a_0+\sum\nolimits_{t=0}^{T} p_t(y_t-c_t) &\geq 0 \\ \underbrace{\sum\nolimits_{t=0}^{T} p_t c_t}_{\text{use of funds}} &\leq \underbrace{a_0+\sum\nolimits_{t=0}^{T} p_t y_t}_{\text{source of funds}} \end{aligned}$$

We conclude that solving the asset difference equation and employing $a_{T+1} \geq 0$ leads to the same type of present value lifetime budget constraint (PVBC) as we encountered in the two-period case.

The dynamic optimization problem

$$\max U = \sum_{t=0}^{T} \beta^t u(c_t)$$

subject to

$$\begin{aligned}\frac{1}{1+r_t}a_{t+1}+c_t &= y_t+a_t \qquad t=0,1,...,T\\ a_{T+1} &\geq 0\end{aligned}$$

can be solved with the help of *discrete time optimal control*.[5] Note that there is one control variable (c_t) and one state variable (a_t) and one exogenous variable (y_t). The following Lagrangian can be formed

$$L=\sum_{t=0}^{T}\beta^t u\left(c_t\right)+\sum_{t=0}^{T}\Lambda_t\left[y_t+a_t-\frac{1}{1+r_t}a_{t+1}-c_t\right]+\Omega_{T+1}a_{T+1}$$

The first order conditions take the form

$$\begin{aligned}\frac{\partial L}{\partial c_t} &= \beta^t\frac{\partial u\left(c_t\right)}{\partial c_t}-\Lambda_t=0 \qquad t=0,1,...,T\\ \frac{\partial L}{\partial a_{t+1}} &= -\frac{1}{1+r_t}\Lambda_t+\Lambda_{t+1}=0 \qquad t=0,1,...,T\\ \frac{\partial L}{\partial a_{T+1}} &= -\frac{1}{1+r_T}\Lambda_T+\Omega_{T+1}=0\\ \frac{\partial L}{\partial\left(\beta^t\lambda_t\right)} &= y_t+a_t-\frac{1}{1+r_t}a_{t+1}-c_t=0 \quad t=0,1,...,T\end{aligned}$$

plus the complementary slackness condition. Combining the first two optimality conditions we get

$$\begin{aligned}\frac{1}{1+r_t}\beta^t\frac{\partial u\left(c_t\right)}{\partial c_t} &= \beta^{t+1}\frac{\partial u\left(c_{t+1}\right)}{\partial c_{t+1}}\\ &\rightsquigarrow \quad \frac{\partial u\left(c_t\right)}{\partial c_t}=\left(1+r_t\right)\beta\frac{\partial u\left(c_{t+1}\right)}{\partial c_{t+1}}\end{aligned}$$

which is nothing else than the generalization of the familiar *intertemporal Euler equation*.

3.3 Digression: The infinite time horizon case with exogenous income

Consider the optimal consumption choice over time for an infinitely lived individual who can freely borrow or lend. Its (time-separable) life-time utility is given

[5] Please note: What follows only serves for illustrative purposes. You need not be familiar with optimal control or any kind of intertemporal optimization methods. The only thing you have to notice is that in the case of $T>2$ we also end up with an intertemporal Euler equation. (The interested student might wonder how discrete time optimal control works in detail. Please compare the supplementary material on this subject.)

by

$$U = \sum_{t=0}^{\infty} \beta^t u(c_t)$$

The individual's asset, a_t, evolves through time according to the following *accumulation constraint*

$$\frac{1}{1+r_t} a_{t+1} + c_t = y_t + a_t$$

where r_t denotes the period t interest rate and y_t denotes the individual's period t income (still exogenous). Further, there is a boundary condition $\lim_{t\to\infty} \Lambda_t a_t = 0$.[6] a_0 is given.

This dynamic optimization problem can be solved with the help of *discrete time optimal control*.[7] Note that there is one control variable (c_t) and one state variable (a_t) and one exogenous variable (y_t). We can form the following Lagrangian

$$L = \sum_{t=0}^{\infty} \beta^t u(c_t) + \sum_{t=0}^{\infty} \Lambda_t \left[y_t + a_t - \frac{1}{1+r_t} a_{t+1} - c_t \right]$$

The first order conditions take the form

$$\begin{aligned}
\frac{\partial L}{\partial c_t} &= \beta^t \frac{\partial u(c_t)}{\partial c_t} - \Lambda_t = 0 \qquad t = 0, 1, ..., T \\
\frac{\partial L}{\partial a_{t+1}} &= -\frac{1}{1+r_t} \Lambda_t + \Lambda_{t+1} = 0 \qquad t = 0, 1, ..., T \\
\frac{\partial L}{\partial (\beta^t \lambda_t)} &= y_t + a_t - \frac{1}{1+r_t} a_{t+1} - c_t = 0 \quad t = 0, 1, ..., T
\end{aligned}$$

plus the boundary condition. Combining the first two optimality conditions we again get

$$\frac{\partial u(c_t)}{\partial c_t} = (1 + r_t) \beta \frac{\partial u(c_{t+1})}{\partial c_{t+1}}$$

3.4 The two period case with labor/leisure choice

So far, income was exogenous. Let us now consider the case where the individual's income depends on the individual's decision how to divide time between work, n_t, and leisure, $l_t = \bar{L} - n_t$; for simplicity let us suppose that income is strictly

[6]This boundary condition is called the transversality condition. For a good treatment compare Obstfeld and Rogoff [7], pp. 63-66.

[7]Please note: What follows only serves for illustrative purposes. You need not be familiar with optimal control or any kind of intertemporal optimization methods. The only thing you have to notice is that in the case of $T \to \infty$ we also end up with an intertemporal Euler equation.

increasing in n_t, $y_t = f(n_t)$. The individual still has access to a bond market and, thus, can freely borrow and lend. The individual's momentary utility now should depend on two arguments, c and l (leisure time). It is often specified as follows

$$u(c_t, l_t) = \begin{cases} \frac{1}{1-\sigma}\left[c_t \nu(l_t)\right]^{1-\sigma} - \frac{1}{1-\sigma} & \text{if } \sigma > 0,\ \sigma \neq 1 \\ \ln(c_t) + \ln \nu(l_t) & \text{if } \sigma = 1 \end{cases}$$

where $\nu(\cdot)$ is twice continuously differentiable. If $\sigma = 1$ then concavity requires that the function $\ln \nu(\cdot)$ is increasing and concave; if $\sigma \neq 1$ then $\nu(\cdot)^{1-\sigma}$ must be increasing and concave if $\sigma < 1$ and decreasing and convex if $\sigma > 1$ (compare King and Rebelo [3], 996n69).[8] A common specification of would be l_t^b, $b > 0$. The PVBC is exactly the same as in subsection 3.1.1.

Let us start with the simple case of a parallel upward shift of the production function for periods 1 and 2. Think of parallel shifts that do not change the schedule for labor's marginal product. For given amounts of work, n_1 and n_2, the real sources of funds (the RHS of the PVBC) increase; this represents a pure wealth effect. Thus, the individual takes more of everything that yields utility; c_1, c_2, l_1 and l_2 increase - the levels of work in both periods, n_1 and n_2, decline.

Next let's consider a proportional upward shift of the production function for periods 1 and 2, i.e. a mix of substitution and wealth effects. On the one hand, as in the Robinson Crusoe economy, consumption becomes less costly relative to leisure - this is true in both periods. That is, consumption rises equally in both periods and leisure decreases. On the other hand, the individual takes more of everything that yields utility - c_1, c_2, l_1 and l_2 increase. *Total effect*: There is an increase in c_1 and c_2 while the effect on l_1 and l_2 is ambiguous (depends on which of the effects is larger).

What happens as r rises? Suppose that savings initially is zero, i.e. the increase in r has no wealth effect and there is a *pure intertemporal substitution effect*. Because the individual can obtain more units of consumption next period for each unit of current consumption forgone, the cost of next period's consumption relative to that of current consumption is lowered. Thus, c_1 falls, and saving rises. The same argument holds for leisure. If the individual takes leisure in period 2, he or she discounts the loss in output, $f(n_2)$, by the factor $(1 + r)$. Therefore, when r rises, the leisure from period 2 becomes cheaper relative to that in period 1. The conclusion is that an increase in the interest rate motivates the individual to substitute toward next period's leisure and away from this period's. Equivalently, this period's work, n_1, rises relative to next period's, n_2. Note also that the increase in n_1 reinforces the effects of the decrease in c_1 in raising current saving.

[8] In Summary 3, we will learn more about why this form is convenient.

3.5 The infinite horizon case with labor/leisure choice

Finally let us consider the optimal consumption and leisure choice over time for an infinitely lived individual who can freely borrow or lend. The only difference compared to the two-period case is that now we make the distinction between *temporary* changes in the production function (and, hence, the individual's labor income) and *permanent* changes in the production function (and, hence, the individual's labor income). A temporary effect only lasts for one period. The production function then shifts back to its original position. A permanent effect means that the change in the production function lasts for ever.

3.5.1 Permanent parallel upward shift of the production function

Suppose there is a permanent parallel upward shift of the production function (while $\{r_t\}_{t=0}^{\infty}$ - which is exogenous - remains unchanged).

- There is a pure wealth effect. Since the long-term average of income increases significantly (upward shift is permanent) the wealth effect is large. Consumption increases significantly in each and every period. Leisure increases significantly in each and every period. Labor decreases substantially at each date.

- Since consumption and income (given by $f(n_t)$) increase at roughly the same amount in the current period, there is no change in saving. Marginal propensity to consume (i.e., the effect of a change in income or output on consumption) is about one.

3.5.2 Temporary parallel upward shift of the production function

Suppose there is a temporary parallel upward shift of the production function (while $\{r_t\}_{t=0}^{\infty}$ remains unchanged).

- There is a **pure wealth effect**. Since the long-term average of income remains almost unaffected, the wealth effect is small. The individual spreads his or her extra income over consumption in all periods. (Consumption increases slightly and equally in all periods. Equivalently, labor supply decreases slightly and equally in all periods.)

- Consumption in the first period increases slightly whereas income is temporarily high; to raise future consumption the individual has to raise current saving. The marginal propensity to consume is small.

3.5.3 Permanent proportional upward shift of the production function

Suppose there is a permanent proportional upward shift of the production function (while $\{r_t\}_{t=0}^{\infty}$ remains unchanged).

- A proportional upward shift of the production function induces both wealth and substitution effects. Long-term average of income increases significantly.
- Wealth effect: The wealth effect is large. Consumption and leisure increase significantly in each and every period; work effort decreases at each date.
- **Static substitution effect between leisure and consumption**: Consumption becomes less costly relative to leisure at each date. Consumption rises equally in all periods, leisure decreases, work effort increases.
- The overall effect on consumption is positive whereas the overall effect on labor supply is ambiguous. Since consumption and income increase at roughly the same amount in the current period there is no change in saving.

3.5.4 Temporary proportional upward shift of the production function

Suppose there is a temporary proportional upward shift of the production function while $\{r_t\}_{t=0}^{\infty}$ remains unchanged.

- Again, we observe a mix of wealth effect and substitution effect. Because the proportional upward shift of the production function is short-lived, the long-term average of income increases by a small amount.
- Wealth effect: The wealth effect is small. Consumption and leisure increase slightly in each and every period; work effort decreases slightly at each date.
- Static substitution effect between leisure and consumption: Current leisure becomes more costly relative to consumption in the current period (period t). Current consumption and current work effort rises, current leisure decreases.
- **Intertemporal substitution effect between leisure today and leisure tomorrow (new element)**: The price of leisure in a given period relative to the price of leisure in the following periods is measured in terms of forgone output (recall that the individual has to give up output when he or she wants to take one unit more leisure): As the marginal productivity of labor in period t increases temporarily, leisure at period t becomes more expensive relative to leisure at period $t + 1$ (and all subsequent periods).

Accordingly, the individual substitutes away from current leisure toward future leisure. Equivalently, current work effort rises and future work effort decreases.

- Total effect: Consumption increase slightly and equally in all periods; it increases a little bit more in the current period. Current work effort rises significantly; future work effort decreases slightly and equally. Current period saving increases. The marginal propensity to consume is small.

3.5.5 Increase in $\{r_t\}_{t=0}^{\infty}$

Suppose the whole future sequence of r_t, $\{r_t\}_{t=0}^{\infty}$, shifts up (while the production function remains unchanged). Moreover, suppose that savings initially is zero, such that the increase in $\{r_t\}_{t=0}^{\infty}$ has no wealth effect.

- For each unit of consumption forgone the individual can get more consumption in the following periods. Hence, an increase in $\{r_t\}_{t=0}^{\infty}$ lowers the costs of future period's consumption relative to that of current consumption; current consumption falls, future consumption rises. What about the labor-leisure choice? Equation (**??**) tells us that an individual discounts future streams of income. The same is true for future losses of income, namely the output loss which arises when an individual takes leisure. Therefore, as r_t rises future leisure becomes cheaper relative to current leisure; current leisure drops, future leisure increases; current labor supply increases, future work effort decreases. Current saving rises.

4 A preliminary version of the RBC model (no investment, no capital)

The next step towards our aim of developing a real business cycle model is to take the theory of consumption demand and labor supply studied so far and add up the actions of all households and firms in a economy.

4.1 Assumptions

- We consider a closed economy with no government sector.
- The economy is populated by infinitely lived households with identical preferences over consumption, c, and leisure, l.[9]

[9] Due to the assumption that all households are identical with respect to both preferences and endowments, we could - if we wanted - work with a *representative household.* As a corollary we would identify individual variables with national aggregates - and in fact would not care about this distinction.

- We do not distinguish firms from households. Each household owns its own business and provides the labor input for this business (we call this a *backyard* or *yeoman farmer economy*).
- There is only one type of physical goods that households/firms can produce, buy, and exchange.
- This good, which cannot be stored, is produced according the technology $y_t = f(n_t)$. $f(\cdot)$ is strictly increasing in its argument, is concave, twice continuous differentiable, and homogenous of degree one.[10]
- There are two markets, a commodity market and a bond market:
  - After every period households/firms sell all of their output on the commodity market at price p (usually normalized to 1); housholds/firms buy these goods for consumption.
  - Each household has access to a bond market and can freely borrow and lend. All bonds are alike and have maturity of one year. Thus, the interest rate is the same for all bonds.
- All markets are always cleared, i.e. total output equals total consumption and each dollar lent by someone on the bond market corresponds to a dollar borrowed by someone else.
- In the aggregate the stock of bonds equals zero (i.e. savings and borrowings equal zero) in each and every period. This means that in the aggregate this economy has no opportunity to save or dissave and that there is no wealth effect due to changes in B.
- There is only one source of shocks in this economy: changes in the production function.

4.2 Graphical representation

Figure 3 contains a graphical representation of the preliminary RBC model. This representation is useful to study the impact effects of a supply shock on the aggregate variables of interest, that is the effect of a shock that happens in period t on time-t variables. We adopt Barro's notation: Capital letters denote aggregate quantities. Y stays for aggregate output, C for aggregate consumption. In contrast to Barro, we use the letter N (instead of L) for work effort and r (instead of R) for the real interest rate (R denotes the nominal interest rate and will be introduced in Section 6).

[10] In fact, $f(\cdot)$ satisfies another important condition called the Inada-condition. Compare Summary 3.

There are two markets in our model economy and, since this is a market-clearing model, there are two aggregate-clearing conditions: (i) Total production equals total uses of output. (ii) Each dollar lent by someone on the credit market corresponds to a dollar borrowed by someone else, that is, in the aggregate $B^d = 0$. (Recall that this is a closed economy.) By Walras' law we explicitly feature only one market, the goods market. We write out the condition for clearing the commodity market during a given period as

$$Y^s \left(\underset{+}{r_t}, ... \right) = C^d \left(\underset{-}{r_t}, ... \right)$$

The LHS of this market clearing condition shows the positive intertemporal substitution effect from the real interest rate on the aggregate supply of good. This is because the aggregate quantity of output is a function of aggregate labor supply ($Y_t^s = F(N_t)$). As we have seen in the previous section an infinitely lived individual with access to a bond market rises current work effort as real interest rate increases. We conclude that in the aggregate current labor supply and, thus, current output increases as r rises. The omitted variables in the function, denoted by ..., include the wealth and substitution effects that arise from changes in the production function. (When any of these elements change the effects on commodity supply show up as a shift of the curve.)

We have seen in the previous section that an infinitely lived individual with access to a bond market lowers current consumption as real interest rate increases. We conclude that in the aggregate demand, C^d, decreases as r rises. Again, the omitted variables in the function, denoted by ..., include the wealth and substitution effects that arise from changes in the production function. Once we know the market clearing level of aggregate output we can compute the corresponding level of aggregate work effort by $N^* = F^{-1}(Y^{s*})$.

Figure 3: Market clearing:
Barro [1], Figure 5.1

4.3 Impact effects of a negative supply shock

Next we investigate the impact effect of a negative supply shock on our market-clearing economy. In reality all endogenous quantities in our market-clearing model are determined simultaneously. However, we derive the effects on Y, C, N, and r step by step.

4.3.1 Temporary parallel downward shift of the production function (Figure 4)

- Primary effect

- The primary effect of a temporary parallel downward shift of the production function in period t is a significant shift of the Y^s curve to the left: For any given real interest rate, more output can be produced.

- Wealth effect

  - We know that a parallel downward shift of the production function involves a pure negative wealth effect. Since the adverse shock is short-lived, the decline in wealth must be small. This means that the C^d curve is shifted a little bit to the left. Moreover, current aggregate leisure decreases slightly and, equivalently, current aggregate work effort, N, increases slightly.
  - The increase in aggregate work effort implies a rise in good supplied which involves a small shift of the Y^s curve to the right (offsets only a small part of the initial cutback in aggregate supply).

- Effects of a change in r

  - Since the shift of the supply curve is larger than that of the demand curve there is *excess demand* for commodities at the initial interest rate, r^*. In other words, at the going interest rate everybody would like to reduce his or her saving or borrow more. This is not possible, since aggregate saving must end up being zero in every period. r must rise to clear the commodity and bond markets; at $(r^*)' > r^*$ the markets clear.
  - The increase in r lowers the quantity of goods demanded *along* the new C^d-curve (current aggregate demand decreases) and raises the quantity supplied *along* the new Y^s-curve (current aggregate supply increases).

- Given $(Y^{s*})'$ we can determine $(N^*)'$ (see above). The story reads as follows: The small decline in wealth raises work effort; the rise in r (intertemporal substitution effect!) reinforces the wealth effect; the aggregate work effort rises.

FIGURE 4: TEMPORARY DOWNWARD SHIFT:
BARRO [1] FIGURE 5.4

4.3.2 Temporary proportional downward shift of the production function (Figure 4)

The only new element compared to the previous case are the substitution effects implied by a change in the marginal product of labor, MPN.

- Primary effect
  - The primary effect of a temporary proportional downward shift of the production function in period t is a significant shift of the Y^s curve to the left.
- Effect on labor effort
  - Since the adverse shock is short-lived, the decline in wealth must be small. This means that the C^d curve is shifted a little bit to the left. Moreover, current aggregate leisure decreases slightly and, equivalently, current aggregate work effort, N, increases slightly.
  - The increase in aggregate work effort implies a rise in goods supplied which involves a small shift of the Y^s curve to the right (offsets only a small part of the initial cutback in aggregate supply).
- Substitution effects:
  - **Static substitution effect between leisure and consumption**: Current leisure becomes cheaper relative to consumption in the current period (period t). Current consumption and current work effort decrease, current leisure rises.
  - **Intertemporal substitution effect between leisure today and leisure tomorrow**: As the marginal productivity of labor in period t decreases temporarily, leisure at period t becomes cheaper relative to leisure at period $t+1$ (and all subsequent periods). Accordingly, households substitute toward current leisure and away from future leisure. Equivalently, current work effort decreases and future work effort rises.
  - The substitution effect between leisure and consumption as well as the intertemporal substitution effect lead to a reduction in current work effort and, hence, to a reduction of goods supplied (leftward shift of the Y^s-curve). Households spread this fall in today's income over consumption and leisure in all periods; current aggregate demand decreases, but only by a small amount (small leftward shift of the C^d-curve).
  - We conclude that the two substitution effects do not alter the general configuration of the commodity market graph given in Figure 4. The only qualitative difference in the results concerns the behavior of work effort which is now *ambiguous*.
- Effect of a change in r

- As before, the shift of the supply curve is larger than that of the demand curve. Hence, there is *excess demand* for commodities at the initial interest rate, r^*. In other words, at the going interest rate everybody would like to reduce his or her saving or borrow more. This is not possible, since aggregate saving must end up being zero in every period. r must rise to clear the commodity and the bond market; the markets clear at $(r^*)' > r^*$.
- The increase in r lowers the quantity of goods demanded *along* the new C^d-curve (current aggregate demand decreases) and raises the quantity supplied *along* the new Y^s-curve (current aggregate supply increases).

4.3.3 Permanent proportional downward shift of the production function (Figure 5)

- Primary effect
  - The primary effect on Y^s in period t remains the same as above.
- Wealth effect
  - When the proportional downward shift is permanent, the wealth effect is big. There is now a strong negative effect on consumer demand (strong leftward shift of the C^d-curve) and, equivalently, a strong positive effect on labor supply.
  - The large increase in work effort implies a significant raise in goods supplied which involves a sizeable shift of the Y^s curve to the right (offsets a significant part of the initial cutback in aggregate supply).
  - Overall C^d and Y^s are reduced by comparable amounts so that r does not change; the markets clear at $(r^*)' = r^*$. (All the effects stemming from a change in r above are nil.)
- Substitution effects
  - The static substitution effect between leisure and consumption lead to a reduction in work effort and, hence, to a reduction of goods supplied (leftward shift of the Y^s-curve) in all periods. This decline in income leads to a decline in consumption (and leisure) in all periods; current aggregate demand decreases (leftward shift of the C^d-curve). We conclude that this substitution effect does not alter the general configuration of the commodity market graph given in Figure 5.
  - Note that there is no intertemporal substitution effect since all periods are affected in the same way!

- Although r does not change both C^d and Y^s fall. The behavior of work effort remains ambiguous.

FIGURE 5: PERMANENT DOWNWARD SHIFT:
BARRO [1] FIGURE 5.7

4.4 A note on dynamics

So far we've been asking ourselves what the impact effects of a productivity shock on our model economy are. But at times we might wonder about the effects of a shock on the variables of interest not only in the time period in which the shock happens (in period t, namely), but also in the subsequent time periods $t+1$, $t+2$, etc. In the present setting, the dynamic responses to a productivity shock are pretty dull: When the shock is temporary, the economy returns to its initial equilibrium in period $t+1$ and stays there forever. When the shock is permanent, the economy reaches a new equilibrium in period t and stays there from time t onwards.

5 Digression: Decentralized labor market

So far we have thought of households as being firms or, as we have called them alternatively, self-employed yeoman farmers. This may seem strange, but as we will show in this section this device is just a useful simplification. The results can easily be extended to the case of decentralized labor market, where we have labor supplying households on one side of the market and labor demanding firms (owned by households) on the other side of the market. We adopt Barro's notation with the exception that (as hitherto) l stands for leisure and n stands for work effort.

5.1 Assumptions

- We introduce firms which are owned by households. All firms produce according to the same CRTS technology. (Given our assumption of constant returns, we can think of output as being produced by a single representative firm that behaves competitively.)

- We introduce a third market, the labor market.

- All labor services are identical. All jobs have identical working conditions. Information is symmetric and there are no externalities and no imperfections whatsoever.

- The labor market is always cleared, i.e. employment equals the labor force and vacancies and unemployment are always zero.

- We neglect distributional effects.

5.2 The equilibrium

Demand side: From the classical microeconomic theory of the firm we know that if the firm acts to benefit its owners then it sets its demand for labor, n^d, to maximize profit in each period. To maximize profit a firm expands employment up to the point at which the value of the marginal product equals the wage rate, i.e. until $p \cdot MPN = w$. An upward shift in the schedule for labor's marginal product leads to a greater quantity of labor demanded at any given real wage rate. Employment expands. (Note that the interest rate does not affect the demand for labor.) We can summarize results by writing down a function for the aggregate demand for labor

$$N^d = N^d(\underset{-}{w/P}, ...)$$

Supply side: In contrast to the previous model households sell their labor services at the real wage rate, w/P, rather than working on their own production. The real wage rate therefore indicates the terms on which people can substitute consumption for leisure. For workers, the real wage takes the place of the marginal product of labor as the measure of how much additional consumption can be gained from more work. If the real wage rises, a worker can get more consumption than before for additional work. Thus, we assume that an increase in the real wage raises the quantity supplied of labor and the demand for consumption. The effects of a change in interest rate, r, remains the same as in the previous model; an increase in the interest rate rises the supply of labor. We can summarize the results by writing down functions for aggregate labor supply

$$N^s = N^s(\underset{+}{w/P}, \underset{+}{r}, ...)$$

where (...) refers to any elements that generate departures of expected future real wage rates from the current value.

Clearing the labor market: Including the labor market in our model gives another aggregate consistency condition: For a given value of interest rate, r, and for a given state of production, the labor market clears when the real wage rate is $(w/P)^*$ and the quantity of work is N^*.

5.3 Conclusions

The presence of the labor market does not alter the analysis and results of section 4, i.e. it does not affect the results about how changes in the production function affect the economy. The only difference is that the presence of an explicit labor market improves the economy's efficiency and performance by equalizing workers' marginal product of labor. At any point in time, each worker receives the same real wage rate because we assumed that all labor services are identical. Labor's marginal product therefore end up being the same on all production processes. This, of course, is efficient; otherwise the economy's total output could rise without changing total work effort, simply by shifting workers from one place to another. On the other hand, there is no reason for workers' marginal products to be equal if a labor market does not exist.

6 Investment and the baseline RBC model

With this section we reach the final step in constructing an RBC model. The most crucial modification with respect to the preliminary version is the introduction of *investment* and *capital*. In the baseline RBC model, capital is simply stored consumption goods. The storage of consumption (we call it "capital accumulation") allows a closed economy to save and dissave. However, dissaving is - as we shall see - possible only to a limited degree.

6.1 Firm investment behavior with exogenous interest rates

We start by investigating the demand for investment, based on a neoclassical firm's investment decision in a perfect foresight setting. That is, we consider the investment decision of an isolated firm which has access to a bond market. The intuition gained in this subsection carries over to the stochastic baseline RBC model with which we are going to deal further below.

When it comes to the decision whether some amount of capital should be purchased or not, a producer faces two alternatives (recall that we assume perfect foresight):

a) Either he or she buys an additional unit of capital on the capital goods market at the beginning of a given period, utilizes it during the period and sells what is left of it at the end of the period on the capital goods market.

$$-P_t + P_{t+1}MPK_t + P_{t+1} - \delta P_{t+1}$$

$(-P_t)$ is the price of an additional unit of capital at the beginning of a given period t. If we hold labor fixed, $P_{t+1} \cdot MPK_t$ is the nominal value of what the producer can produce and hence sell additionally when he or she purchases an additional unit of capital and utilizes it during the current period to produce one unit of output (there is only one type of good in the model and this good is used for either consumption or investment). $P_{t+1} - \delta P_{t+1}$ is what the producer gets when she sold the utilized capital good at the end of period at P_{t+1} minus what the value of depreciation at $t + 1$.

b) Or he or she invests the same amount of money which the purchase of a unit of capital would have required in a risk free bond which yields some risk free (*nominal*) return

$$-P_t + P_t\left(1 + R_t\right)$$

$-P_t$ is the price of an additional unit of capital at the beginning of a given period t which now is invested in a risk free bond. At the end of the period the producer/invester gets principal plus interest, $P_t\left(1 + R_t\right)$.

For the producer to be indifferent between the two alternatives, both must yield the same return. Conversely, if one of the two alternatives is more attractive, the producer will choose this alternative and will continue to do so until the two alternatives are equivalent. The resulting equation is particularly meaningful when we rewrite it as follows

$$\begin{aligned} -P_t + P_{t+1}MPK_t + P_{t+1} - \delta P_{t+1} &= -P_t + P_t\left(1 + R_t\right) \\ P_{t+1}\left(MPK_t + 1 - \delta\right) &= P_t + P_t R_t \end{aligned}$$

$$P_{t+1}MPK_t = P_t R_t + \delta P_{t+1} + \left[P_t - P_{t+1}\right] \tag{5}$$

In words: The firm should equate the value marginal product of capital to an *implicit rental price* that consists of three terms: the interest cost, the depreciation cost, and the capital loss cost.

Let us express this condition in real terms. We start by modifying it to

$$\begin{aligned} P_{t+1}\left(MPK_t + 1 - \delta\right) - P_t &= P_t R_t \\ \frac{P_{t+1}\left(MPK_t + 1 - \delta\right) - P_t}{P_t} &= R_t \end{aligned}$$

Then we define

$$\begin{aligned} \pi_t &\equiv \frac{P_{t+1} - P_t}{P_t} \\ &\rightsquigarrow P_{t+1} = P_t\left(1 + \pi_t\right) \end{aligned}$$

Thus,

$$\begin{aligned} \frac{P_t\left(1 + \pi_t\right)\left(MPK_t + 1 - \delta\right)}{P_t} &= 1 + R_t \\ \left(1 + \pi_t\right)\left(MPK_t + 1 - \delta\right) &= 1 + R_t \end{aligned}$$

Moreover, let us define

$$\begin{aligned} 1 + r_t &\equiv \frac{\left(1 + R_t\right)}{\left(1 + \pi_t\right)} \\ &\rightsquigarrow \left(1 + R_t\right) = \left(1 + \pi_t\right)\left(1 + r_t\right) \end{aligned}$$

We end up with

$$\begin{aligned}(1+\pi_t)(MPK_t + 1 - \delta) &= (1+\pi_t)(1+r_t) \\ MPK_t - \delta &= r_t\end{aligned} \tag{6}$$

This optimality condition says that period t investment should continue to the point at which its marginal return net of depreciation is the same as the real return on a bond.

FIGURE 6: INVESTMENT AND CAPITAL:
BARRO [1], FIGURE 9.6

Depicting the optimality condition (see Figure 6) helps to recognize that the condition (6) implicitly defines the desired stock of capital of the firm, $\hat{k}_t$. The effects of a change in MPK_t, r_t or δ, respectively, on $\hat{k}_t$ can be read off the graph. (Do this experiment!) We conclude that

$$\hat{k}_t = \hat{k}\left(\underset{-}{r_t}, \underset{-}{\delta}, ...\right) \tag{7}$$

Suppose the *capital accumulation equation* of the firm is given by

$$k_t = i_t + (1-\delta)\, k_{t-1}$$

where it is *gross investment*. Substituting $\hat{k}_t$ for k_t in the capital accumulation equation yields a demand function for gross investment, i_t^d

$$\begin{aligned} i_t^d &= \hat{k}_t - (1-\delta)\, k_{t-1} \\ i_t^d &= \hat{k}\left(\underset{-}{r_t}, \underset{-}{\delta}, ...\right) - (1-\delta)\, k_{t-1} \\ i_t^d &= i^d\left(\underset{-}{r_t}, \underset{?}{\delta}, \underset{-}{k_{t-1}}, ...\right)\end{aligned}$$

Note that the effect of a change in δ on i_t^d is ambiguous.

6.2 Assumptions of the baseline RBC model

After having studied the optimal investment decision by an individual producer, we now continue by setting up a full-fledged one-good, one-shock RBC model including capital and investment. We continue to think of producers as self-employed yeoman farmers who invest with the goal of maximizing their lifetime utility. We could, however, show that the results would be exactly the same when we explicitly allowed for markets in labor and capital. We make the following assumptions:

- We consider a simple, closed economy without a government sector.

- The economy is populated by infinitely lived households with identical preferences over consumption, c, and leisure, l.[11]

- We do not distinguish firms from households, i.e. we do not explicitly deal with firms or factor markets.

- There is only one type of good that people produce and exchange.

- This good is produced according to the technology $y_t = f(k_{t-1}, n_t)$; k stands for *physical capital* (i.e. producer's durable equipment and structures). $f(\cdot)$ is strictly increasing in its argument, is concave, twice continuous differentiable, and homogenous of degree one.[12] Note that it takes time to make new capital operational; for use in production during period t, the stock from period $t-1$ is available. Further note that we abstract from variations in the utilization rate of capital, i.e. the capital stock equals capital services (in terms of a standard shift per day).

- After every period producers sell all of their output on the commodity market at price p; housholds/firms buy these goods either for consumption or investment. The initial labelling choice as consumables or capital is irreversible; households cannot consume their capital at a later date. (They can, however, allow capital to depreciate and not replace it. See below.)

- There is a single type of capital, i.e. there is no difference between old and new capital.

- The moving of capital goods from one production activity to another is possible at negligible costs; each unit of capital must therefore end up with the same physical marginal product.

- Capital goods tend to depreciate over time; the amount of depreciation during period t is a constant fraction of capital, $\delta \in [0,1]$. The capital stock that is available in period $t+1$ is therefore given by[13]

$$k_t = i_t + (1-\delta)\, k_{t-1}$$

[11]Due to the assumption that all households are identical with respect to both preferences and endowments, we could - if we wanted - work with a *representative household*. As a corollary we would identify individual variables with national aggregates - and in fact would not care about this distinction.

[12]In fact, $f(\cdot)$ satisfies another important condition called the Inada-condition. Compare Summary 3.

[13]Note that the capital accumulation equation usually looks slightly different, $k_{t+1} = i_t + (1-\delta)\, k_t$. This difference has to do with the specification of technology which is usually given by $y_t = f(k_t, n_t)$.

We implicitly assume that firms adjust their capital stock costlessly.[14]

- We assume a given initial capital stock for this economy (usually the steady state level).
- As usual there is a bond market. All bonds are alike and have maturity of one year. Thus, the interest rate is the same for all bonds.
- All markets are always cleared, i.e. total output equals total consumption and each dollar lent by someone on the bond market corresponds to a dollar borrowed by someone else.
- In the aggregate the stock of bonds equals zero (i.e. savings and borrowings equal zero) in each and every period. This means that the only way for this closed economy to shift resources through time is by accumulating/depreciating capital.
- There is only one source of shocks in this economy: changes in the production function.

6.3 Graphical representation

The graphical representation of the baseline RBC model contained in Figure 7 is useful to study the impact effects of a productivity shock on the variables of interest. It is less useful, however, to understand the dynamic responses of the model economy. The problem is, that - in contrast to the preliminary RBC model - investment increases in response to a positive productivity shock (i.e., the representative household optimally saves some fraction of the higher current output). An rise of the capital stock, of course, has consequences on the dynamics of the model economy. Nevertheless, we start by studying Figure 7.

Figure 7: Full-fledged RBC model:
Barro [1], Figure 9.7

As in the preliminary RBC model there are two markets and, since this is a market-clearing model, there are two aggregate-clearing conditions: (i) Total production equals total uses of output; (ii) Each dollar lent by someone on the credit market corresponds to a dollar borrowed by someone else. Thus, in the aggregate $B^d = 0$. (Recall that this is a closed economy.)

By Walras' law we explicitly feature only one market, the goods market. We write out the condition for clearing the commodity market during a given period as

$$Y^s\left(\underset{+}{r_t}, ...\right) = C^d\left(\underset{-}{r_t}, ...\right) + I_t^d\left(\underset{-}{r_t}, ...\right)$$

[14] For the case where capital is costly to install compare e.g. Romer [8], chapter 8 or Obstfeld and Rogoff [7], section 2.5.2.

As in the preliminary version, the LHS of this market clearing condition shows the positive intertemporal substitution effect from the real interest rate on the aggregate supply of good. The omitted variables in the function, denoted by ..., include the wealth and substitution effects that arise from changes in the production function as well as the quantity of capital from the previous period.

The RHS contains the two components of aggregate demand. Consumption demand is exactly the same as in the RBC without investment and capital. Accordingly, C^d depends negatively on r_t. I_t^d represents aggregate gross investment demand[15]

$$I_t^d = \hat{K} - (1 - \delta) K_{t-1}$$

As we have seen in the subsection on firm's demand for investment, i_t^d depends negatively on r_t. We conclude that in the aggregate gross investment also depends negatively on r_t. Again, the omitted variables in the function, denoted by ..., include the wealth and substitution effects that arise from changes in the production function as well as the quantity of capital from the previous period and the depreciation rate. Note that the investment demand curve is *less* negatively sloped that the consumption demand curve since it is especially sensitive to variations in the real interest rate.[16]

6.4 Impact effects of a positive supply shock

6.4.1 Temporary parallel upward shift of the production function (Figure 8)

FIGURE 8: TEMPORARY UPWARD SHIFT:
BARRO [1], FIGURE 9.8

- Effects on the Y^s and the Y^d curve (shifts of the curves):
  - The improvement of the production function in period t shifts the **aggregate supply** curve to the right.
  - Wealth rises, but by only a small amount, because the increase in income is temporary. Therefore
    - **Consumer demand** rises by a small amount.

[15]Note that aggregate gross investment cannot be negative whereas aggregate net investment, $I_t^d - \delta K_{t-1} = \hat{K}_t - K_{t-1}$, can be negative but only up to the amount of depreciation. Aggregate real saving, finally, is given by $\hat{K}_t - K_{t-1}$.

[16]Generally, because net investment is a small fraction of the existing capital stock, small percentage changes in the desired capital stock generate large changes in net investment demand.

        * **Work effort** falls by a small amount. This decrease in work effort offsets part of the increase in the supply of goods (small leftward shift of Y^s).[17]
    - **Gross investment demand** does not shift since there is no change in MPK_t. Note that any effects of changes in the real interest rate are captured by movements *along* the Y^d curve.

- Effects on real interest rate (movements along the curve):
    - At the initial market clearing interest rate, the quantity of goods supplied *exceeds* the quantity of good demanded. Equivalently, we can say that the disturbance creates an excess of desired real saving over investment demand

$$(S \equiv)\, Y^s - C^d > I^d$$

      Hence, the **real interest rate** must fall for the commodity market to clear. This has the following consequences:
        * As we move from the situation of the excess supply to the new equilibrium point we note that the fall in the real interest rate raises **consumption** and **investment demand**.
        * A cut in r not only lowers the costs of current consumption; the same argument holds true for leisure. Thus, **work effort** falls short of the initial amount.

- Closer inspection of the quantitative responses of consumption demand, work and investment demand:
    - Because the wealth effect is weak, the changes in consumption and work will be large only if there is a substantial decline in the real interest rate.
    - Since investment demand is highly responsive to r, a small decrease in r is sufficient to clear the commodity market. Hence, most of the rise in output reflects an increase in investment and there is only a small change in consumption and work.

6.4.2 Temporary proportional upward shift of the production function

Upward shift of the schedule for labor's marginal product, MPN_t (Figure 8)

[17] Note that such a feedback only happens when labor supply changes for reasons other than a change in the real interest rate (here because of a pure wealth effect).

- The improvement of the schedule for labor's marginal product, MPN_t, shifts the **aggregate supply** curve to the right.
- Wealth rises, but by only a small amount, because the increase in income is temporary. Therefore
  - **Consumer demand** rises by a small amount.
  - **Work effort** falls by a small amount. This decrease in work effort offsets part of the increase in the supply of goods (small leftward shift of Y^s).
- The change in MPN_t motivates people to raise **work effort** (current leisure becomes more expensive relative to consumption - static substitution effect).
- Because the improvement in productivity is temporary, there is also an intertemporal-substitution effect (current leisure becomes more expensive relative to future leisure), which reinforces the tendency for work to rise.
- The increase in work effort implies additional rightward shifts to commodity supply. Output expansion becomes even larger.
- At the initial market clearing interest rate, the quantity of goods supplied *exceeds* the quantity of goods demanded. Hence, the **real interest rate** must fall for the commodity market to clear:
  - As we move from the situation of the excess supply to the new equilibrium point we note that the fall in the real interest rate raises **consumption** and **investment demand**.
  - A cut in r not only lowers the costs of current consumption; the same argument holds true for leisure. Thus, **work effort** falls short of the initial amount.

Upward shift of the schedule for capital's marginal product, MPK_t (Figure 9)

FIGURE 9: TEMPORARY UPWARD SHIFT OF MPK_t

- Recall that this marginal product refers to the output for period $1 + t$.
- To keep things simple, assume, for the moment, that no changes occur to the production function in period t. That is we neglect the positive effects on the current supply of goods, Y^s, and the effects from increased wealth. Given this assumption, the only effect from the disturbance is an increase in net investment demand (since MPK_t increases) which leads to a rightward shift of the Y^d curve.

- This leads to an increase of the real interest rate and, hence, output. As we move along the curves to the new equilibrium we note
  - that the higher real interest rate leads to a fall in investment and consumption demand. As usual the reaction of investment demand on a increase in r_t is bigger than that of consumption demand. Note that the net effect on investment demand still is positive.
  - that the expansion of output reflects the positive effect of the higher real interest rate on work effort (movement along Y^s).
- When we relax the assumption that no changes occur to the production function in period t, then the effect of a temporary proportional upward shift of MPK_t is a combination of shifts of the schedule for capital's and labor's marginal product. For a discussion compare Subsection 6.5 and Set 9, Problem 11.

6.4.3 Permanent parallel upward shift of the production function

- Effects on the Y^s and the Y^d curve (shifts of the curves):
  - When the favorable shock is permanent, the wealth effects become important.
    * There is a strong positive effect on consumer demand (large forward shift of the demand curve).
    * There is a strong negative effect on work effort (implies a backward shift of the supply curve which is larger than before).
  - Overall, the rightward shift in demand roughly equals that in supply. Therefore, output rises, *but the real interest rate does not change.*

6.4.4 Permanent proportional upward shift of the production function

Compare subsection 6.5, Dynamic response of the baseline RBC model to productivity shocks. In particular, observe the *large positive effect on the real interest rate*!

6.5 Dynamic response to a productivity shock

So far we've been asking ourselves what the impact effects of a positive productivity shock on our model economy are. But we might wonder about the effects of a shock on the variables of interest not only in the time period in which the shock happens (in period t, namely), but also in the subsequent time periods $t+1$, $t+2$, etc. We call this the dynamic response of a model economy. Those dynamics are represented by means of so-called *impulse response functions* (IRF).

King and Rebelo [3] generate the IRF for the baseline RBC as it is presented in this summary. They consider a proportional upward shift of the production function. First, they let the shock be *very short-lived*. Next, they consider the case where the shock is *fairly long-lived*.[18] Finally they consider the case of a *permanent* shock. In the remainder of this subsection, we simply quote the relevant part from King and Rebelo [3]. The IRF for all three cases are included in the PDF file containing the figures to this summary. (If you do not immediately understand what is going on: do not become desperate. It *is* a complex story going on here. Just stick at it.)

Purely temporary shock: "Productivity is assumed to increase by one percent in the initial period (date 1). Given the rise in the marginal product of labor resulting from the increase in productivity, the representative household faces an unusually high opportunity cost of taking leisure in this initial period. While there are offsetting income and substitution effects, the model's preferences were chosen so that a permanent increase in the real wage (such as the one associated with the trend in technical progress) generates exactly offsetting income and substitution effects so that labor and leisure are left unchanged. An implication of this result is that N has to rise in response to a temporary productivity increase. With a temporary shock, there is a much smaller income effect and there is greater incentive to substitute intertemporally, since the current wage is high relative to expected future wages. On net, the positive labor response amplifies the productivity shock. Half of this response is due to the direct effect of the productivtiy shock and half due to the increase in labor.

"The representative agent must choose what the economy will do with all this additional output. One possibility is to consume it all in period one. However, this would be inefficient given that the marginal utility of consumption is decreasing, thus inducing a preference for smooth consumption paths. It is optimal to increase consumption both today and in the future. In fact, given that there are many future periods, only a small fraction of the output windfall will be consumed at time 1; most of it will be invested. Thus investment rises by 8% in response to a 2% increase in output. It is interesting to note that the high volatility of investment, which Keynes ascribed to 'animal spirits', arises naturally in this economy as the flip side of consumption smoothing.

"In the future, which begins with period 2, productivity returns to its original level. The only difference relative to period zero is that the economy has accumulated some capital and only a relatively small amount since the productivity shock lasted just for one period. The optimal policy for the economy is to gradually reduce this excess capital by enjoying higher levels of consumption and leisure. The real interest rate signals individuals to adopt these consumption and

[18] We call this a *persistent* shock; productivity shocks persist for a while but not forever.

leisure paths: with a purely temporary change in productivity, the real interest rate falls in the impact period and in all future periods, making it desirable for individuals to choose consumption profiles that decline through time toward the steady-state level."

Long-lived (but not permanent) shock: "The same mechanisms are at work as in the case of a purely temporary shock, but these effects are now drawn out over time. We now have an extended interval in which productivity is above normal . During this interval, workers respond by increasing their labor supply and most of the additional output is invested. Interestingly, high productivity is now initially associated with high real interest rate, since the marginal product of capital schedule, $MPK_t - \delta$, is shifted upwards by the productivity shock and by a higher level of future labor input, with capital responding only gradually via the accumulation of investment. However, later in the impulse responses, the rate of return is below its steady-state level because the capital stock has been built up while the stimulative effects of the productivity shock and labor input have dissipated. This leads consumption to initially grow through time and then subsequently to decline back toward the stationary level. Later in the impulse responses, as productivity converges slowly to its normal level, labor supply actually drops below the steady-state level as the economy enters a phase that resembles the transitional dynamics discussed above. Investment also eventually drops below the steady state, as the economy runs down the capital that was accumulated during the initial expansion.

"As with the case of the purely temporary shock discussed above, the early part of the impulse responses is dominated by the fact that the productivity shock raises the desirability of work effort, production, investment and consumption; the latter part of the impulse response function is dominated by the transitional dynamics, i.e., reduction of capital back toward its stationary level."

Permanent shock: "Figure 10 shows that the impulse responses of all variables are substantially affected by changing the driving process parameter ρ from 0.979 to 1. Part of this difference involves the fact that a permanent productivity shock leads to an identical, proportional, long-run increase in consumption, investment and output, while the stationary shock has no long-run effect. There are also important differences in how the economy responds in the short-run depending on the value of ρ. (...) In the general equilibrium of our RBC model, there is one additional channel [other than wage and wealth effects]: *interest rate effects* that induce intertemporal substitutions for consumption and leisure. In general, these intertemporal price effects are a powerful influence, but one that is not much discussed in informal expositions of the comparative dynamics of RBC models. In particular, permanent increases in productivity lead to high real interest rates and these induce individuals to substitute always from 0 consumption and leisure (..)."

"We are now in a position to describe why a permanent shift in productivity (arising when $\rho = 1$) has a smaller effect on labor than a persistent but ultimately temporary shocks ($\rho = 0.979$). When the shock is temporary, there is a small wealth effect that depresses labor supply but temporarily high wages and real interest rates induce individuals to work hard. When the shock is permanent, there are much larger wealth effects and the pattern of intertemporal substitution in response to wages is reversed since future wages are high relative to current wages. However, labor still rises in this case in response to productivity shocks due to very large intertemporal substitution effects of interest rates."

6.6 Evaluation of the model

Figure 10: Cyclical behavior of the baseline RBC model (taken from King and Rebelo [3], Table 3)

How well does the baseline RBC model match empirical business cycles? One way of evaluating the predictions of the baseline RBC model is to compare moments that summarize the actual experience of an economy with similar moments from the model.[19] If we assume a long-lived but not permanent shock then the model's predictions fit fairly well the various business-cycle facts (compare Figure 10); the summary statistics reported e.g. by King and Rebelo [3], Table 3, are comparable to those discussed in Summary 1. However, there are also evident discrepancies. Notably, consumption and labor input in the baseline model are each much less volatile than in the data. Further, the baseline RBC model produces a strongly procyclical real wage and real interest rate, which does not accord well with the U.S. experience.

References

[1] Barro, Robert J. (1997), Macroeconomics, 5th edition, MIT Press

[2] Hansen, Gary D., and Randall Wright (1992), The labor market in real business cycle theory, Federal Reserve Bank of Minneapolis Quarterly Review, Spring

[3] King, Robert G. and Sergio T. Rebelo (1999), Resuscitating real business cycles, in Handbook of Macroeconomics, Volume 1, edited by J.B. Taylor and M. Woodford, Elsevier Science B.V.

[4] King, Robert G. and Mark W. Watson (1998), The solution of singular linear difference systems under rational expectation, International Economic Review, Vol. 39, No. 4, November

[19] An alternative way of evaluating the predictions of the basic RBC model would be to generate time series by simulating the model with the innovations to the actual Solow residual and to compare the simulated time series with the actual ones.

[5] Landsburg, Steven T. (1993), The armchair economist

[6] Mas-Colell, Andreu, Michael D. Whinston and Jerry R. Green (1995), Microeconomic Theory, Oxford University Press

[7] Obstfeld, Maurice and Kenneth Rogoff (1996), Foundations of international macroeconomics, The MIT Press

[8] Romer, David (1996), Advanced Macroeconomics, The McGraw-Hill Companies

[9] Uhlig, Harald (1997), A Toolkit for Analyzing Nonlinear Dynamic Stochastic Models Easily