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Lecture 1

1 A SIMPLE, ONE SECTOR, STOCHASTIC GROWTH MODEL

1.1 The economic environment

- A single good, single input (capital), representative agent model
- Who operates in this economy, what objectives they have, what opportunities (constraints) they face and how they go about to satisfy their objectives

A representative agent

- Objectives (preferences):

$$U = E_0 \sum_0^{\infty} \beta^t u(c_t) \quad 0 \leq \beta \leq 1 \tag{1}$$

$$u(c_t) = \frac{1}{1-\gamma} c_t^{1-\gamma} \quad \gamma \geq 0 \tag{2}$$

- Opportunities (production):

$$f(k_t) = z_t k_t^{\alpha} \quad 0 < \alpha \leq 1 \tag{3}$$

  - Physical capital depreciates at the rate of δ per period
  - Stochastic technology:

$$z_{t+1} = z_t^{\rho} v_{t+1} \qquad v_{t+1} \quad \text{iid} \quad 0 \leq \rho \leq 1 \tag{4}$$

  - Information: v_t becomes known in period t

- Mode of behavior: Rational pursuit of objectives.

  Rational expectations

1.2 Optimization

In the absence of distortions there is an equivalence between the solution to the problem faced by the social planner and the solution obtained in a competitive equilibrium. It saves on notation to use a fictitious social planner who maximizes the utility of a representative agent subject to the resource constraint of the economy

In particular, the value function is

$$V(k_t, z_t) = \max\{u(c_t) + \beta E_t V(k_{t+1}, z_{t+1}) \tag{5}$$

where E_t is conditional expectation taken in period t and k_t and z_t are the state variables. The resource constraint is

$$f(k_t) = c_t + i_t \tag{6}$$

The capital stock evolves as follows

$$k_{t+1} = (1-\delta)k_t + i_t \tag{7}$$

Combining (6) and (7) we have

$$k_{t+1} = (1-\delta)k_t + f(k_t) - c_t \tag{8}$$

The objective is to maximize (5) subject to (32) and (31) by selecting the optimal sequence $\{c_t, k_{t+1}\}_{t=0}^{\infty}$. Let λ_t be the Langrange multiplier associated with (32). The FOC (Euler equations) are

$$u_{ct} = \lambda_t \tag{9}$$

$$\beta E_t V_{kt+1} = \lambda_t \tag{10}$$

Noting that $V_{kt+1} = \lambda_{t+1}(f_{kt+1} + 1 - \delta)$ and combining (9) and (10) leads to

$$\beta E_t u_{ct+1}(f_{kt+1} + 1 - \delta) = u_{ct} \tag{11}$$

Equations (31), (32) and (33) describe the dynamics of the system. Substituting from (2) and (3) we arrive at a system of three nonlinear, stochastic first order difference equations. As this system cannot be solved explicitly we need to consider solutions to a tractable, approximate version of it[1]. There exist several ways of carrying out this approximation (grid, projection method,...). For the time being we will restrict attention to a simple linear approximation around the deterministic steady state of the model. Let us write (31), (32) and (33) as

$$g_0(z_t, z_{t+1}, v_{t+1}) = 0 \tag{12}$$

$$g_1(c_t, k_t, k_{t+1}, z_t) = 0 \tag{13}$$

$$E_t g_2(c_t, c_{t+1}, k_{t+1}, z_{t+1}) = 0 \tag{14}$$

The first thing we need to do is to calculate the deterministic steady state of the system. That is the values $k = k_t = k_{t+1}$ and $c = c_t = c_{t+1}$ with $z_t = z_{t+1} = z = 1$ that satisfy (12), (13) and (14). It can be easily verified that

$$k = (\frac{\beta^{-1} + \delta - 1}{\alpha})^{\frac{1}{\alpha - 1}}$$

and

$$c = (k)^\alpha - \delta k$$

Taking a (log) first order Taylor approximation of (12), (13) and (14) around k, c and z we have

$$z'_{t+1} = \rho z'_t + v'_{t+1} \tag{15}$$

$$A1c'_t + A2k'_t + A3k'_{t+1} + A4z'_t = 0 \tag{16}$$

$$B1c'_t + E_t B2c'_{t+1} + B3k'_{t+1} + E_t B4z'_{t+1} = 0 \tag{17}$$

where the A's and B's are functions of $\alpha, \beta, \delta, \gamma, \rho, k, c, z$ and thus known numbers, and $c'_t = (c_t - c)/c$ and so on. Equations (15,16,17) form a system of three stochastic, first order, linear difference equations.

[1] Another way of dealing with non-linearity is to do the approximation before taking the FOCs. Either start with (1) and (2) but proceed to approximate them by a quadratic polynomial for (1) and a linear function for (2). Or assume from the beginning a quadratic utility and linear production functions. In either case the FOCs are linear.

The agents only need information on k_t and z_t in order to decide how much to consume and invest in period t. k and z are the state variables of the system (the former is endogenous while the latter is exogenous). It is then *reasonable* to assume that their decisions take the form (if we are wiling to rule out sunspots, bubbles)

$$k'_{t+1} = h_1(k'_t, z'_t) \qquad c'_t = h_2(k'_t, z'_t)$$

Our objective is to determine the functions h_1 and h_2. There are several ways of doing so and hence for solving for the variables of interest, c'_t and k'_{t+1}.

1.3 Solution

1.3.1 I. The method of undetermined coefficients

This is a simple method. It may become cumbersome-infeasible if the system is too large because it may require the simultaneous solution of many high order polynomials. For fairly small systems it usually works fine (sometimes one may have to use some tricks to reduce the order of the system to be solved).

Let us assume that

$$h_1(k'_t, z'_t) = e1k'_t + e2z'_t \tag{18}$$

$$h_2(k'_t, z'_t) = e3k'_t + e4z'_t \tag{19}$$

(there is no constant term because everything is measured as deviations from the steady state). Substituting (18) and (19) into (16) and (17) and noting that $E_t c'_{t+1} = E_t h_{2,t+1}$ and $E_t z'_{t+1} = rz'_t$ we arrive at

$$M1k'_t + M2z'_t = 0 \tag{20}$$

$$M3k'_t + M4z'_t = 0 \tag{21}$$

where the M_j s will in general be polynomial functions of the parameters of interest, $e1, e2, e3, e4$. For (20) and (21) to hold for all values of k' and z' it is required that $M1 = M2 = M3 = M4 = 0$. We thus have four equations in four unknowns that can be solved to determine the policy functions $h_1$and h_2.

1.3.2 II. The method of R. Farmer

Consider again the linearized system of equations (15,16,17). This system can be written as

$$\begin{pmatrix} r & 0 & 0 \\ A4 & A2 & A1 \\ 0 & 0 & B1 \end{pmatrix} \begin{pmatrix} z_t' \\ k_t' \\ c_t' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & A3 & 0 \\ B4 & B3 & B2 \end{pmatrix} \begin{pmatrix} z_{t+1}' \\ k_{t+1}' \\ c_{t+1}' \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & B4 & 0 & B2 \end{pmatrix} \begin{pmatrix} v_{t+1}' \\ w_{zt+1}' \\ w_{kt+1}' \\ w_{ct+1}' \end{pmatrix} \tag{22}$$

where $w_{ct+1}' = E_t(c_{t+1}') - c_{t+1}'$(and so on) or, writing it in a more compact form

$$R1s_t' = R2s_{t+1}' + R3n_{t+1}' \tag{23}$$

If $R1$ is non singular then we multiply (23) through by $R1^{-1}$ to arrive at

$$s_t' = W1s_{t+1}' + W2n_{t+1}' \tag{24}$$

Vaughan (1970) has developed a method for solving equations such as this one. It involves transforming $W1$ into Jordan canonical form. In particular, $W1$ can be written as

$$W1 = CJC^{-1} \tag{25}$$

where J is a diagonal matrix with elements the eigenvalues of $W1$ and C is the matrix whose columns are the eigenvectors of $W1$. The roots (eigenvalues) of $W1$ determine the type of solution. Let m be the number of state (”predetermined”) variables in the system (in our case there are two, k and z). If the number of eigenvalues that lie outside the unit circle is equal to m then a unique, rational expectations, stationary solution exists; if it is greater than m then there exist multiple stationary equilibria; and if it is smaller than m then no stationary solution exists. Hence, in our case we need two roots outside and one inside the unit circle to have a unique equilibrium. Suppose that the parameters of the model are such that this is indeed the case.

Let us use (25) in (24), multiply through by C^{-1} and then take expectations. The transformed system now takes the form

$$x_t = JE_tx_{t+1} \tag{26}$$

where the first element (row) of $x,$ say, $x1$, is simply the product of the first row of C^{-1} times the vector column (z_t', k_t', c_t') and so on. Note that the element (row) of x_t that corresponds to the eigenvalue that lies inside the unit circle has a very interesting property. Namely, it has to be equal to zero because -as can be easily seen by recursively substituting for the expectation in the RHS- the corresponding right hand side is equal to zero. But each element of x defines a linear relationship between c_t', k_t' and z_t'. Subsequently,we have managed to get a solution for c' in terms of k' and z'. Substituting this solution into (16) we get a solution for k_{t+1}'. Voila.

III. The Blanchard-Khan method

This is the same method as that employed by Farmer. Substituting equation (15) into (16,17) and making sure that we have the state variables at the top and then the endogenous variables produces the following system

$$\begin{pmatrix} A3 & 0 \\ B3 & B2 \end{pmatrix} \begin{pmatrix} k_{t+1}' \\ E_t c_{t+1}' \end{pmatrix} = \begin{pmatrix} A2 & A1 \\ 0 & B1 \end{pmatrix} \begin{pmatrix} k_t' \\ c_t' \end{pmatrix} + \begin{pmatrix} A4 \\ rB4 \end{pmatrix} z_t' \tag{27}$$

Or written more compactly -after having multiplied by the inverse of the first matrix on the LHS-

$$\begin{pmatrix} k_{t+1}' \\ E_t c_{t+1}' \end{pmatrix} = P \begin{pmatrix} k_t' \\ c_t' \end{pmatrix} + P1 z_t' \tag{28}$$

The solution method involves writing P in its canonical form $P = C^{-1}JC$, calculating the eigenvalues and eigenvectors of P, decomposing J and C so that the eigenvalues outside the circle (hopefully one in our case, in order to have a unique solution, note how eq. 28 differs from eq. 24) appear at the top and then using equation (2) from Blanchard and Khan to compute the solution for c_t' and k_{t+1}'.

A complication arises when the system does not consist of first order difference equations. In such a case the method of Farmer (or Blanchard and Khan or King, Plosser and Rebello) is not directly applicable. One must then first reduce the order of the system by suitable transformations (such as defining lagged values as new variables and so on). Note though that while such reduction may be possible most of the time, it is sometimes infeasible.

For instance, suppose that you solved the maximization problem by substituting (32) (for c_t)and (31) in (5) and then linearized around the steady state. The resulting solution would take the form of a linear second order difference equation

$$Y1k_t' + Y2k_{t+1}' + Y3E_t k_{t+2}' + Y4z_t' = 0 \tag{29}$$

wher Y_i are constants. Equation (29) can be written in the form of equation (27) (or equivalently, (22)) by introducing a new variable $x_{t+1}' = k_{t+2}'$

$$\begin{pmatrix} Y2 & Y3 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} k'_{t+1} \\ E_t x'_{t+1} \end{pmatrix} = \begin{pmatrix} -Y1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} k'_t \\ x'_t \end{pmatrix} + \begin{pmatrix} -Y4 \\ 0 \end{pmatrix} z'_t \tag{30}$$

Another complication arises when the matrix on the LHS of 27 is singular. In this case, one cannot multiply by its inverse to get the system into the form 28. There are several alternatives in this case:

a) Carry out successive substitutions in the system of equations until this matrix has become non-singular (get rid of some of the variables).

b)Use methods that rely on generalized eigenvalue decomposition (rather than the Jordan form). Klein, JEDC provides a description (as well as the matlab codes for this solution method).

1.4 Using *dynare* to solve the system

Recall that our system consists of equations 31, 32 and 33:

$$z_{t+1} = z_t^{\rho} v_{t+1} \tag{31}$$

$$k_{t+1} = (1-\delta)k_t + f(k_t) - c_t \tag{32}$$

$$\beta E_t u_{ct+1}(f_{kt+1} + 1 - \delta) = u_{ct} \tag{33}$$

Using the particular functional forms for production and utility postulated above and taking logs in eq. 31 gives

$$log z_{t+1} - \rho log z_t - log v_{t+1} = 0 \tag{34}$$

$$c_t + k_{t+1} - (1-\delta)k_t - z_t k_t^{\alpha} = 0 \tag{35}$$

$$c_t^{-\gamma} - \beta c_{t+1}^{-\gamma}(z_{t+1}\alpha k_{t+1}^{\alpha-1} + 1 - \delta) = 0 \tag{36}$$

Dynare prefers a somewhat different timing convention, with k_t representing the choice variable rather than the state variable in t. That is,

$$log z_t - \rho log z_{t-1} - log v_t = 0 \tag{37}$$

$$c_t + k_t - (1-\delta)k_{t-1} - z_t k_{t-1}^{\alpha} = 0 \tag{38}$$

$$c_t^{-\gamma} - \beta c_{t+1}^{-\gamma}(z_{t+1}\alpha k_t^{\alpha-1} + 1 - \delta) = 0 \tag{39}$$

We typically take log-linear approximations so that the deviations from the steady state are expressed in percentage terms. The file $model\_1.mod$ generates the solution (the file $model\_1\_sym.m$ offers an equivalent solution using Klein's matlab code).