# The basic RBC model

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# 1 Model specification

This section specifies the structure of the economy.

### 1.0.1 The household

The household's objective is to maximize expected life-time utility :

$$E_0[\sum_{t=\infty} \beta^t U(C_t, l_t)] \tag{1}$$

where $0 < \beta < 1$ is a constant discount factor, $C_t$ denotes consumption and $l_t$ leisure. The expectation operator $E_t[.]$ is conditional expectation in period $t$ given the available information in $t$, $\Omega_t$, that is $E_t[X_{t+1}] = E[X_t|\dot{\Omega}_t]$. $U(C_t, l_t)$ is a utility function, increasing and concave in both of arguments. The following specific utility function will be used:

$$U(C_t, l_t) = \frac{1}{1-\sigma}(C_t^{\mu}(l_t)^{1-\mu})^{1-\sigma} \tag{2}$$

The household faces the following time constraint:

$$1 = l_t + h_t \tag{3}$$

where $h_t$ is the amount worked. In each and every period the household faces the following budget constraint which equates the uses to the sources of funds:

$$w_t h_t + r_t K_t + \Pi_t = C_t + I_t + T_t \tag{4}$$

where $w_t$ is the real wage rate in period $t$, $K_t$ is the amount of physical capital [1] at the beginning of period $t$, $r_t$ is the rental rate of capital, $\Pi_t$ are the household's share of the profits of the representative firm, $C_t, I_t$ and $T_t$ denote consumption, investment and (lump-sum) taxes respectively. Physical capital accumulates according to

$$K_{t+1} = (1-\delta)K_t + I_t \tag{5}$$

where $0 \leq \delta \leq 1$ denotes the rate of depreciation.

[1] It is assumed that the capital is owned by the households and is rented out to the firms. This makes the maximization problem of the firm completely static, a useful simplification.

### 1.0.2 The representative firm

The representative firm produces a consumption good $Y$. The firm maximizes profits

$$\Pi_t = Y_t - w_t h_t - r_t K_t \tag{6}$$

subject to the available technology:

$$Y_t = A_t (K_t)^{\alpha} (\Gamma_t h_t)^{1-\alpha} \tag{7}$$

$\Gamma_t$ represents Harrod neutral deterministic technical progress that evolves according to $\Gamma_t = \gamma \Gamma_{t-1}$ where $\gamma \geq 1$.

### 1.0.3 The government

The government has a balanced budget in each and every period

$$T_t = G_t \tag{8}$$

where $G$ is government spending.

### 1.0.4 The shocks

There are two types of shocks in the model. The first one, $\varepsilon_t^A$, is a technology shock. It evolves according to the following equation

$$A_{t+1} = \overline{A} A_t^{\rho_A} \varepsilon_{t+1}^A \tag{9}$$

where $\rho_A$ denotes the autocorrelation parameter of $A$. The second type of shock, $\varepsilon_t^G$, is a public spending shock and is described by the following equation

$$G_{t+1} = \overline{G} G_t^{\rho_G} \varepsilon_{t+1}^G \tag{10}$$

again, with $\rho_G$ denoting the autocorrelation parameter of $G$.

## 1.1 First order conditions

### 1.1.1 The household

In period t the representative agent solves the following problem

$$\max_{C_t,h_t,I_t,K_{t+1}} \sum_{t=O}^{\infty} \beta^t \frac{1}{1-\sigma}(C_t^{\mu}(1-h_t)^{1-\mu})^{1-\sigma} \tag{11}$$

subject to the sequence of budget constraints given by equations 4 and 5. Let $\Lambda_t^B$ and $\Lambda_t^B$ be the Langrange multipliers associated with these two budget constraints in period t. The first order conditions are:

$$\begin{array}{lrcl} C_t: & \mu C_t^{\mu(1-\sigma)-1}(1-h_t)^{(1-\mu)(1-\sigma)} & = & \Lambda_t^B \\ h_t: & (1-\mu)C_t^{\mu(1-\sigma)}(1-h_t)^{\sigma(1-\mu)-1} & = & \Lambda_t^B w_t \\ I_t: & \Lambda_t^B & = & \Lambda_t^K \\ K_{t+1}: & \Lambda_t^K & = & \beta E_t\{\Lambda_{t+1}^B r_{t+1} + \Lambda_{t+1}^K(1-\delta)\} \end{array} \tag{12}$$

### 1.1.2 The firm

The firm selects the quantity of labor and capital to hire in order to maximize profits

$$\Pi_t = \max_{(h_t,K_t)} A_t(K_t)^{\alpha}(\Gamma_t h_t)^{1-\alpha} - w_t h_t - r_t K_t \tag{13}$$

The first order conditions of the firm are[2].

$$(1-\alpha)A_t K_t^{\alpha}(\Gamma_t h_t)^{1-\alpha} = w_t h_t \tag{14}$$

$$\alpha A_t K_t^{\alpha}(\Gamma_t h_t)^{1-\alpha} = r_t K_t \tag{15}$$

[2] Note that there is a single good in the model which serves as the numeraire, that is its price is set to unity

### 1.1.3 Market clearing

Using the standard national accounts identity $Y_t = C_t + I_t + G_t$ together with the definition of the production function, $Y$, produces the good's market clearing condition

$$A_t(K_t)^{\alpha}(\Gamma_t h_t)^{1-\alpha} = C_t + I_t + G_t \tag{16}$$

# 2 Making the model stationary

Recall that there is sustained growth in the model, $\Gamma_t = \gamma\Gamma_{t-1}$ with $\gamma \geq 1$. We can then "deflate" each growing variable and generate the corresponding stationary version by dividing it by $\Gamma$. That is, $x_t = X_t/\Gamma_t$ with $X_t \in \{C_t, I_t, K_t, ...\}$. The stationary representation of the Langrage multipliers can then be derived by plaguing $X_t = x_t\Gamma_t$ in the first order conditions. It turns out that $\Lambda_t^i = \lambda_t^i\Gamma_t^{\mu(1-\sigma)-1}$for $i = K, B$. In what follows we assume that $\gamma = 1$ in order to simplify the analysis.

We now report the stationary representation of the equilibrium of this economy. It includes the first order conditions of the households and the firms, the market clearing conditions, the law of motion of capital and the behavior of the two shocks. Note that a lower case letter denoted the stationary value of any variable as described above

$$\mu c_t^{\mu(1-\sigma)-1}(1-h_t)^{(1-\mu)(1-\sigma)} = \lambda_t^B \tag{17}$$

$$(1-\mu)c_t^{\mu(1-\sigma)}(1-h_t)^{(1-\sigma)(1-\mu)-1} = \lambda_t^B w_t \tag{18}$$

$$\lambda_t^K = \lambda_t^B(\equiv \lambda_t) \tag{19}$$

$$\beta E_t\{\gamma^{\mu(1-\sigma)-1}\lambda_{t+1}^B r_{t+1} + (1-\delta)\gamma^{\mu(1-\sigma)-1}\lambda_{t+1}^K\} = \lambda_t^K \tag{20}$$

$$(1-\alpha)a_t k_t^{\alpha} h_t{}^{-\alpha} = w_t \tag{21}$$

$$\alpha a_t k_t^{\alpha-1} h_t{}^{1-\alpha} = r_t \tag{22}$$

$$y_t = a_t k_t^{\alpha} {h_t}^{1-\alpha} \tag{23}$$

$$\gamma k_{t+1} = (1-\delta) k_t + i_t \tag{24}$$

$$a_t k_t^{\alpha} {h_t}^{1-\alpha} = c_t + i_t + g_t \tag{25}$$

$$\log(a_{t+1}) = \log(A) + \varrho_A \log(a_t) + \log(\varepsilon_{t+1}^{A}) \tag{26}$$

$$\log(g_{t+1}) = \log(G) + \varrho_G \log(g_t) + \log(\varepsilon_{t+1}^{G}) \tag{27}$$

# 3 The steady-state

The equations describing the equilibrium of the economy are nonlinear, so it may not be feasible to solve them analytically. Such systems are typically solved by first log-linearly approximating around the steady state and then solving the resulting log-linear equations. We need then to determine the behavior of the system in the steady-state. We will represent the steady-state value of any variable using an asterisk (*) and dropping the time index. For instance, the steady state value of consumption is $c^* = c_t = c_{t-1} \ldots$

### 3.0.4 The steady state conditions

Dropping the time subscripts and using asterisks in the equations describing the equilibrium of the economy (equations 17 -27) we have

$$\mu (c^*)^{\mu(1-\sigma)-1} (1-h^*)^{(1-\mu)(1-\sigma)} = \lambda^* \tag{28}$$

$$(1-\mu)(c^*)^{\mu(1-\sigma)-1} (1-h^*)^{\sigma(1-\mu)-1} = \lambda^* w^* \tag{29}$$

Dividing (28) by (29)

$$w^* = \frac{1-\mu}{\mu} \frac{c^*}{1-h^*} \tag{30}$$

$$r^* = (\frac{1}{\beta_0} - 1 + \delta) \text{ with the definition } \beta_0 \equiv \beta\gamma^{\mu(1-\sigma)-1} \tag{31}$$

$$(1-\alpha)a^*(\frac{k^*}{h^*})^\alpha = w^* \tag{32}$$

$$\alpha a^*(\frac{k^*}{h^*})^{\alpha-1} = r^* \tag{33}$$

$$(\gamma + \delta - 1)k^* = i^* \tag{34}$$

$$a^*(k^*)^\alpha(h^*)^{1-\alpha} = c^* + i^* + g^* \tag{35}$$

#### 3.0.5 Calibration

We determine the steady-state values[3] by using the following numerical values assigned to the structural parameters: $\alpha = 0.35$, $\beta = 0.99$, $\delta = 0.025$, $\gamma = 1.007$, $\mu = 0.3$, $\sigma = 2$, $\rho_A = 0.95$, $\rho_G = 0.95$, $a = 1$.

# 4 Log-linearization

As mentioned earlier we will approximate the nonlinear system of equations characterizing the equilibrium of this economy with a system consisting of log-linear ones. The strategy is to take a first order Taylor approximation around the steady state

In general, the linearization around the steady state of a given equation can be done as follows. First, we approximate both sides of the equal sign with a first order Taylor series :

$$\begin{aligned} f(x) &= g(y) \\ f(x^*) + f'(x^*)\underbrace{(x - x^*)}_{dx} &= g(y^*) + g'(y^*)\underbrace{(y - y^*)}_{dy} \end{aligned}$$

[3]In the literature, we often work the other way around. That is, we use average values for the variables under consideration (for instance, $k/y$) as their steady state values and then use the steady state equations to solve for the parameters values.

Second, we devide by the steady state equation $f(x^*) = g(y^*)$

$$\begin{aligned}
\frac{f(x^*)}{f(x^*)} + \frac{f'(x^*)}{f(x^*)}(x - x^*) &= \frac{g(y^*)}{g(y^*)} + \frac{g'(y^*)}{g(y^*)}(y - y^*) \\
\frac{f'(x^*)}{f(x^*)}(x - x^*) &= \frac{g'(y^*)}{g(y^*)}(y - y^*) \\
\frac{f'(x^*)\,x^*}{f(x^*)}\frac{(x - x^*)}{x^*} &= \frac{g'(y^*)\,y^*}{g(y^*)}\frac{(y - y^*)}{y^*}
\end{aligned}$$

where we define $\frac{x-x^*}{x^*} \equiv \log x - \log x^* \equiv \hat{x}$ as the percentage deviation of $x$ from its steady state value (the same is true for $\frac{y-y^*}{y^*}$).

We now give an example of how this approximation can be applied by using the first order condition for consumption (the first line in equation 12)

$$\begin{aligned}
& \mu[\mu(1-\sigma) - 1](c^*)^{\mu(1-\sigma)-2}(1-h^*)^{(1-\mu)(1-\sigma)}(c_t - c^*) \\
& -(c^*)^{\mu(1-\sigma)-1}(1-\mu)(1-\sigma)(1-h^*)^{(1-\mu)(1-\sigma)-1}(h_t - h^*) \\
= \; & (\lambda_t^B - \lambda^{B*})
\end{aligned}$$

$$\begin{aligned}
[\mu(1-\sigma) - 1](c^*)^{-1}(c_t - c^*) - (1-\mu)(1-\sigma)(1-h^*)^{-1}(h_t - h^*)\frac{h^*}{h^*} &= (\lambda_t^B - \lambda^{B*})/\lambda^{B*} \\
(\mu(1-\sigma) - 1)\hat{c}_t + \frac{h^*}{1-h^*}(1-\mu)(1-\sigma)\hat{h}_t &= \hat{\lambda}_t
\end{aligned}$$

Applying this procedure everywhere, we can derive the log-linearized version of our model. Each of these equations will be used to determine a variable according to the following order $\{\hat{c}, \hat{h}, \hat{\lambda}, \hat{w}, \hat{r}, \hat{k}, \hat{\imath}, \hat{y}, \hat{\alpha}, \hat{g}\}$. Notice that the number of equations is equal to the number of unknown variables.

$$(\mu(1-\sigma) - 1)\hat{c}_t + \frac{h^*}{1-h^*}(1-\mu)(1-\sigma)\hat{h}_t = \hat{\lambda}_t \tag{36}$$

$$\mu(1-\sigma)\hat{c}_t + \frac{h^*}{1-h^*}(\sigma(\mu-1) - \mu)\hat{h}_t - \hat{w}_t = \hat{\lambda}_t \tag{37}$$

$$\beta^* r^* \hat{r}_{t+1} = \hat{\lambda}_t - \hat{\lambda}_{t+1} \tag{38}$$

$$\hat{w}_t + \alpha \hat{h}_t = \alpha \hat{k}_t + \hat{a}_t \tag{39}$$

$$(1-\alpha)\hat{h}_t - \hat{r}_t = (1-\alpha)\hat{k}_t - \hat{a}_t \tag{40}$$

$$\frac{i^*}{k^*}\hat{\imath}_t = \gamma \hat{k}_{t+1} - (1-\delta)\hat{k}_t \tag{41}$$

$$(1-\alpha)\hat{h}_t - \frac{c^*}{y^*}\hat{c}_t - \frac{i^*}{y^*}\hat{\imath}_t = \frac{g^*}{y^*}\hat{g}_t - \hat{a}_t - \alpha \hat{k}_t \tag{42}$$

$$\hat{y}_t - (1-\alpha)\hat{h}_t = \alpha \hat{k}_t + \hat{a}_t \tag{43}$$

$$\hat{a}_{t+1} - \varrho_A \hat{a}_t = \hat{\varepsilon}^A_{t+1} \tag{44}$$

$$\hat{g}_{t+1} - \varrho_G \hat{g}_t = \hat{\varepsilon}^G_{t+1} \tag{45}$$

## 4.1 Solution method

This system of the log-linearized equations can be rewritten in the following matrix equation form:

$$M_{CC}\c{C}_t = M_{CS}S_t \tag{46}$$

$$M_{SS0}S_{t+1} + M_{SS1}S_t = M_{SC0}\c{C}_{t+1} + M_{SC1}\c{C}_t + M_{Se}\varepsilon_{t+1} \tag{47}$$

This system can be interpreted as a state-space system[4]. Hence the first –vector– equation is the measurement equation of the system: It contains the static equations of the system and links the control variables, represented by the vector $\c{C}_t$, to the state variables, represented by the vector $S_{t.}$ In our problem, the control

[4]Things are slightly more complicated when some of the difference equations in our model are higher than first order. One must then define some latent variables that will reduced the order of the system to one. See Blanchard and Khan, 1980.

variables vector is given by $Ç_t = \left\{\hat{c}_t, \hat{h}_t, \hat{\imath}_t, \hat{w}_t, \hat{r}_t, \hat{y}_t\right\}'$ whereas the state variables vector is given by $S_t = \left\{\hat{k}_t, \hat{a}_t, \hat{g}_t, \hat{\lambda}_t\right\}'$. Hence, the measurement equation consists[5] of equations (36,37,42,39, 40, 43).

The second vector equation can be interpreted as the state equation. It accounts for the dynamic link between control variables, state variables and the surprises, represented by $E_t = \{E_t[\widehat{x}_{t+1}] - \widehat{x}_{t+1}, \widehat{\varepsilon}_t\}$, for $x \in \{k, a, g, \lambda\}$. The state equation contains the dynamic equations, and in our model, it consists of equations (38, 41, 44 and 45).

It must be emphasized that the state variables are ordered in a very particular order, which is key for the solution method used, as it will become clear in a moment. We first introduce the endogenous backward looking state variables, namely the capital stock , then the exogenous shocks, namely $a_t$ and $g_t$, and finally the forward variables, namely the shadow price $\lambda_t$. This is essential since our method will rest on a partition of matrices, conditioned by the status of the variables. One must be extremely careful in identifying the state and control variables.

The matrices in our model are given as follows (they are the coefficients of the system of equations (36-(45) :

#### 4.1.1 Measurement equation:

$$M_{CC} = \begin{bmatrix} \mu(1-\sigma)-1 & -(1-\mu)(1-\sigma)\frac{h*}{1-h*} & 0 & 0 & 0 & 0 \\ \mu(1-\sigma) & -(\sigma(\mu-1)-\mu)\frac{h*}{1-h*} & 0 & -1 & 0 & 0 \\ -\frac{c*}{y*} & 1-\alpha & -\frac{i*}{y*} & 0 & 0 & 0 \\ 0 & \alpha & 0 & 1 & 0 & 0 \\ 0 & 1-\alpha & 0 & 0 & -1 & 0 \\ 0 & \alpha-1 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{48}$$

[5] The ordering of the control variables and corresponding equations does not have to follow any particular pattern. I chose this one arbitrarily.

$$M_{CS} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ -\alpha & -1 & \frac{g*}{y*} & 0 \\ \alpha & 1 & 0 & 0 \\ 1-\alpha & -1 & 0 & 0 \\ \alpha & 1 & 0 & 0 \end{bmatrix} \tag{49}$$

### 4.1.2 State equation:

$$M_{SS0} = \begin{bmatrix} \gamma & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{50}$$

$$M_{SS1} = \begin{bmatrix} \delta-1 & 0 & 0 & 0 \\ 0 & -\rho_A & 0 & 0 \\ 0 & 0 & -\rho_G & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{51}$$

$$M_{SC0} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & r^*\beta^* & 0 \end{bmatrix} \tag{52}$$

$$M_{SC1} = \begin{bmatrix} 0 & 0 & \frac{i*}{k*} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{53}$$

$$M_{SE} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{54}$$

# 5 Appendix

A description of the Blanchard-Khan method.

The first step is to rewrite the model in a form that only involves state variables. This is done using equation (46) :

$$\mathcal{C}_t = M_{cc}^{-1} M_{cs} \mathcal{S}_t$$

Plugging this expression in (47), we get :

$$\mathcal{S}_{t+1} = W_{\mathcal{S}}\, \mathcal{S}_t + \mathcal{E}_{t+1}$$

where :

$$W_{\mathcal{S}} = -\left(M_{ss0} - M_{sc0}\, M_{cc}^{-1}\, M_{cs}\right)^{-1} \left(M_{ss1} - M_{sc1}\, M_{cc}^{-1}\, M_{cs}\right)$$

Blanchard et Kahn [1980] established that existence and uniqueness of the solution rely on a condition relating the position of eigenvalues of $W_{\mathcal{S}}$ to the unit circle. Let $N_B$ and $N_F$ denote respectively the number of Backward looking state variables and Forward looking state variables. Then let $M_I$ and $M_O$ be the number of eigenvalue Inside and Outside of the unit circle, then :

> *If $N_B = M_I$ and $N_F = M_O$ then there exists a unique path that is the solution of the rational expectations problem, converging to the steady state of the model.*

This configuration corresponds to a saddle path. There exists a unique $\lambda_0$ such that the path $\{\widehat{\lambda}_t\}_{t=0}^{\infty}$ satisfies the transversality condition. In fact, the method we are using tries to find this value $\lambda_0$.

The diagonalization of $W_{\mathcal{S}}$ leads to :

$$W_{\mathcal{S}} = P\, D\, P^{-1}$$

where $D$ is the matrix of eigenvalues and $P$ is the associated eigenvectors matrix. We first sort the eigenvalues by ascending order, and rearrange the eigenvector matrix according to the new order. This leads tot the following partition of matrices $P$ and $P^{-1}$ :

$$P = \begin{pmatrix} P_{BB} & P_{BF} \\ P_{FB} & P_{FF} \end{pmatrix},\ P^{-1} = \begin{pmatrix} P_{BB}^{\star} & P_{BF}^{\star} \\ P_{FB}^{\star} & P_{FF}^{\star} \end{pmatrix}$$

This partition relies on the position of eigenvalues relative to the unit circle. Indeed, a $B$ index means that the corresponding eigenvalue is of modulus less than one, whereas an index $F$ means that the eigenvalue is outside of the unit circle.[6].

We then rewrite the system to obtain a simple diagonal system. We thus define :

$$\widetilde{\mathcal{S}}_t = P^{-1}\mathcal{S}_t$$

so that :

$$P^{-1}\mathcal{S}_{t+1} = P^{-1}W_{\mathcal{S}}P\,P^{-1}\mathcal{S}_t + P^{-1}W_{\mathcal{E}}\mathcal{E}_{t+1}$$

thus :

$$\widetilde{\mathcal{S}}_{t+1} = D\,\widetilde{\mathcal{S}}_t + R\,\mathcal{E}_{t+1}$$

The partition of $R$ is in the lines of that of $P$ :

$$R = \begin{pmatrix} R_{B.} \\ R_{F.} \end{pmatrix}$$

For the state vector :

$$\widetilde{\mathcal{S}}_t = \begin{pmatrix} \widetilde{\mathcal{S}}_{B,t} \\ \widetilde{\mathcal{S}}_{F,t} \end{pmatrix}$$

Thus the law of motion of forward looking variables can be written :

$$\widetilde{\mathcal{S}}_{F,t+1} = D_F\widetilde{\mathcal{S}}_{F,t} + R_{F.}\mathcal{E}_{t+1}$$

The conditional expectation at time $t$ of $\widetilde{\mathcal{S}}_{F,t+1}$ is given by :

$$E_t\widetilde{\mathcal{S}}_{F,t+1} = D_F\widetilde{\mathcal{S}}_{F,t}$$

since $D_F$ is a diagonal matrix, iterating on the process leads to :

$$\widetilde{\mathcal{S}}_{F,t} = \lim_{j\to\infty} D_F^{-j}E_t\widetilde{\mathcal{S}}_{F,t+j}$$

[6]This is why the order in which variables appear in the system is important. When the Blanchard and Kahn conditions are satisfied, the partition corresponds to a decomposition between backward looking and forward looking state variables.

$\widetilde{\mathcal{S}}_{F,t}$ is thus bounded if the limit of $D_F^{-j} E_t \widetilde{\mathcal{S}}_{F,t+j}$ is finite as $j \to \infty$. If the transversality condition is satisfied, we have[7] :

$$\lim_{j \to \infty} D_F^{-j} E_t \widetilde{\mathcal{S}}_{F,t+j} = 0$$

This implies :

$$\widetilde{\mathcal{S}}_{F,t} = P^\star_{FB} \mathcal{S}_{B,t} + P^\star_{FF} \mathcal{S}_{F,t} = 0$$

This equation allows us to define the initial condition to $\mathcal{S}_{F,t}$ which is consistent with the transversality conditions and the initial conditions of the predetermined variables:

$$\mathcal{S}_{F,t} = (P^\star_{FF})^{-1} P^\star_{FB} \mathcal{S}_{B,t}$$

Defining $Q = \left( I \vdots - {P^\star_{FF}}^{-1} P^\star_{FB} \right)'$, we get :

$$E_t \left( \widehat{\mathcal{S}}_{B,t+1} \right) = W_{\mathcal{S},\mathcal{B}.} \, Q \, \widehat{\mathcal{S}}_{B,t} = M_{SS} \, \widehat{\mathcal{S}}_{B,t}$$

Since the surprises on predetermined variables are null[8], the surprises only rely on the innovation of the exogenous shocks. Thus :

$$\widehat{\mathcal{S}}_{B,t+1} = M_{SS} \, \widehat{\mathcal{S}}_{B,t} + M_{SE} \, \widehat{\varepsilon}_{t+1}$$

where $M_{MSE} = R_{B.} \, Q$.

The only thing that we still have to do is to obtain the solution for the measurement equation :

$$\mathcal{C}_t = M_{cc}^{-1} M_{cs} \, Q \, \mathcal{S}_{B,t} = \Pi \, \mathcal{S}_{B,t}$$

[7] Actually this condition corresponds to ruling bubbles out.

[8] By definition, a predetermined variable is such that $E_t \widehat{\mathcal{S}}_{t+1} = \widehat{\mathcal{S}}_{t+1}$