International Monetary Economics: Appendix to Class Note 2 (Multiperiod extension)

1 Definitions

$$ca_t \equiv tb_t - rd_{t-1} \tag{1}$$

$$y_t^P \equiv \frac{r}{1+r}\sum_{j=0}^{\infty}\frac{y_{t+j}}{(1+r)^j} \tag{2}$$

The trade balance in an economy without government spending and capital is

$$tb_t = y_t - c_t \tag{3}$$

2 Model

2.1 Optimization Problem and the Budget Constraint

The (representative) household problem is subject to a sequence of budget constraints

$$\max_{\{x_t\}_{t=0}^{\infty}} \mathbb{E}_t \sum_{j=0}^{\infty} \beta^t u(c_{t+j}) \tag{4}$$

$$y_{t+j} + d_{t+j} = c_{t+j} + (1+r)d_{t+j-1} \tag{5}$$

$d_t > 0$ (< 0) represents external liabilities (assets). The Lagrangian

$$\mathcal{L} = \mathbb{E}_t \sum_{j=0}^{\infty} \beta^t \Big\{ u(c_{t+j}) + \lambda_{t+j} [y_{t+j} + d_{t+j} - c_{t+j} - (1+r) d_{t+j-1}] \Big\} \tag{6}$$

The first order conditions (FOC) with respect to c_t and d_t

$$u'(c_t) = \lambda_t \tag{7}$$
$$\lambda_t = \beta(1+r)\mathbb{E}_t \lambda_{t+1} \tag{8}$$

combine to get the Euler equation

$$u'(c_t) = \beta(1+r)\mathbb{E}_t u'(c_{t+1}) \tag{9}$$

Iterating on the budget constraint[1]

$$(1+r)d_{t-1} = y_t - c_t + d_t \tag{10}$$
$$d_t = \frac{1}{1+r}(y_{t+1} - c_{t+1} + d_{t+1}) \tag{11}$$
$$d_{t+1} = \frac{1}{1+r}(y_{t+2} - c_{t+2} + d_{t+2}) \tag{12}$$
$$\ldots \tag{13}$$
$$(1+r)d_{t-1} = \sum_{j=0}^{s} \frac{y_{t+j} - c_{t+j}}{(1+r)^j} + \frac{d_{t+s}}{(1+r)^s} \tag{14}$$

for $s \to \infty$, and with $r > 0$

$$(1+r)d_{t-1} = \sum_{j=0}^{\infty} \frac{y_{t+j} - c_{t+j}}{(1+r)^j} + \lim_{s\to\infty} \frac{d_{t+s}}{(1+r)^s} \tag{15}$$
$$(1+r)d_{t-1} = \sum_{j=0}^{\infty} \frac{y_{t+j} - c_{t+j}}{(1+r)^j} \tag{16}$$

Relating debt and the trade balance:

[1] Never use expectation operators in this process as the constraints must hold for any arbitrary shock realization, i.e. not only in expectations.

$$(1+r)d_{t-1} = \sum_{j=0}^{\infty} \frac{tb_{t+j}}{(1+r)^j} \tag{17}$$

i.e. positive debt requires (some) periods with a positive trade balance in the future.

The solution for c_t with an explicit (quadratic) utility function and the assumption $\beta(1+r) = 1$: [2]

$$u(c) = -\frac{1}{2}(c - \bar{c})^2 \tag{18}$$

Use the Euler equation

$$-(c_t - \bar{c}) = -\mathbb{E}_t(c_{t+1} - \bar{c}) \tag{19}$$

$$c_t = \mathbb{E}_t c_{t+1} \tag{20}$$

It is also true that $c_{t+1} = \mathbb{E}_{t+1} c_{t+2}$. Combine this with the above expression and use the law of iterated expectations[3]

$$c_t = \mathbb{E}_t c_{t+1} = \mathbb{E}_t \mathbb{E}_{t+1} c_{t+2} \tag{21}$$

$$c_t = \mathbb{E}_t c_{t+2} \tag{22}$$

generalized

$$c_t = \mathbb{E}_t c_{t+1} = \mathbb{E}_t c_{t+2} = \ldots = \mathbb{E}_t c_{t+j} \quad \forall j \tag{23}$$

Use this in the iterated DBC

$$(1+r)d_{t-1} = \sum_{j=0}^{\infty} \frac{y_{t+j} - c_{t+j}}{(1+r)^j} \tag{24}$$

$$(1+r)d_{t-1} = \sum_{j=0}^{\infty} \frac{y_{t+j}}{(1+r)^j} - \sum_{j=0}^{\infty} \frac{c_{t+j}}{(1+r)^j} \tag{25}$$

$$(1+r)d_{t-1} = \sum_{j=0}^{\infty} \frac{\mathbb{E}_t y_{t+j}}{(1+r)^j} - \sum_{j=0}^{\infty} \frac{\mathbb{E}_t c_{t+j}}{(1+r)^j} \tag{26}$$

[2] A solution relates the variable in question (here: c_t) to exogenous and/or predetermined variables only.

[3] $\mathbb{E}(X) = \mathbb{E}(\mathbb{E}(X|Y)$

using that consumption is constant in expectation

$$(1+r)d_{t-1} = \sum_{j=0}^{\infty} \frac{\mathbb{E}_t y_{t+j}}{(1+r)^j} - \sum_{j=0}^{\infty} \frac{c_t}{(1+r)^j} \tag{27}$$

2.2 Deterministic Endowment Process

Because endowment is deterministic

$$(1+r)d_{t-1} = \sum_{j=0}^{\infty} \frac{y_{t+j}}{(1+r)^j} - \sum_{j=0}^{\infty} \frac{c_t}{(1+r)^j} \tag{28}$$

Solve the geometric series

$$\frac{1+r}{r} c_t = \sum_{j=0}^{\infty} \frac{y_{t+j}}{(1+r)^j} - (1+r)d_{t-1} \tag{29}$$

$$c_t = \frac{r}{1+r} \sum_{j=0}^{\infty} \frac{y_{t+j}}{(1+r)^j} - r d_{t-1} \tag{30}$$

Note that the weights given to each period's endowment ($\frac{r}{1+r}\frac{1}{(1+r)^j}$) sum up to unity. In other words, the first term on the right hand side is a weighted average of the (expected) endowments over time. It will be denote by y_t^P.[4]

Solution for the debt level:

$$y_t^P = c_t + r d_{t-1} \tag{31}$$

$$y_t + d_t = c_t + (1+r)d_{t-1} \tag{32}$$

Combine

$$y_t^P - y_t = d_t - d_{t-1} \tag{33}$$

i.e. increase borrowing if your current income is below the weighted average of (expected) endowments over time.

[4]Later we will see that y_t^P is the endowment level that is compatible with $ca_t = 0$

The current account: From the definition of the current account[5]

$$ca_t \equiv tb_t - rd_{t-1} \tag{34}$$
$$ca_t = y_t - (c_t + rd_{t-1}) \tag{35}$$
$$ca_t = y_t - y_t^P \tag{36}$$
$$ca_t = -(d_t - d_{t-1}) \tag{37}$$

Solution for the trade balance:

$$tb_t = ca_t + rd_{t-1} \tag{38}$$
$$tb_t = y_t - y_t^P + rd_{t-1} \tag{39}$$

2.3 Stochastic Endowment Process

Suppose

$$y_t = \rho y_{t-1} + \varepsilon_t \tag{40}$$

with $0 < \rho < 1$ and $\mathbb{E}_t \varepsilon_{t+j} = 0 \ \forall j$. Then

$$c_t = \frac{r}{1+r} \sum_{j=0}^{\infty} \frac{\mathbb{E}_t y_{t+j}}{(1+r)^j} - rd_{t-1} \tag{41}$$
$$c_t = \frac{r}{1+r} \sum_{j=0}^{\infty} \frac{\rho^j y_t}{(1+r)^j} - rd_{t-1} \tag{42}$$

Solve the geometric series

$$c_t = \frac{r}{1+r} \frac{1+r}{1+r-\rho} y_t - rd_{t-1} \tag{43}$$
$$c_t = \frac{r}{1+r-\rho} y_t - rd_{t-1} \tag{44}$$

remember $y_t^P = c_t + rd_{t-1}$, hence

[5]The current account is the trade balance plus net investment income. We take the minus because $d_t > 0$ denotes debt.

$$y_t^P = \frac{r}{1+r-\rho} y_t \tag{45}$$

Solution for the trade balance:

$$tb_t = y_t - y_t^P + rd_{t-1} \tag{46}$$

$$tb_t = y_t - \frac{r}{1+r-\rho} y_t + rd_{t-1} \tag{47}$$

$$tb_t = \frac{1-\rho}{1+r-\rho} y_t + rd_{t-1} \tag{48}$$

Solution for the current account:

$$ca_t \equiv tb_t - rd_{t-1} \tag{49}$$

$$ca_t = \frac{1-\rho}{1+r-\rho} y_t \tag{50}$$

Solution for the debt level:

$$ca_t = -(d_t - d_{t-1}) \tag{51}$$

$$d_t = d_{t-1} - ca_t \tag{52}$$

$$d_t = d_{t-1} - \frac{1-\rho}{1+r-\rho} y_t \tag{53}$$

The debt level is a random walk.

Pro-cyclicalities: How does the current account and the trade balance react to endowment shocks?

$$\frac{\mathrm{d}tb_t}{\mathrm{d}\varepsilon_t} = \frac{1-\rho}{1+r-\rho} > 0 \tag{54}$$

$$\frac{\mathrm{d}ca_t}{\mathrm{d}\varepsilon_t} = \frac{1-\rho}{1+r-\rho} > 0 \tag{55}$$

The trade balance is a procyclical shock absorber. For $\rho = 0$, it reacts almost one for one with the endowment shock. For $\rho \to 1$, it hardly reacts to the endowment shock.[6]

[6] Since the shock is (in the limit) permanent, consumption moves one for one with the endowment shock.

Conuter-cyclicality: For the endowment shock and the current account to have a negative correlation, assume a non-stationary income process

$$\Delta y_t = \rho \Delta y_{t-1} + \varepsilon_t \tag{56}$$

with $0 < \rho < 1$, $\mathbb{E}_t \varepsilon_{t+j} = 0 \ \forall j$, and $\Delta y_t = y_t - y_{t-1}$.

$$ca_t = y_t - y_t^P \tag{57}$$

$$ca_t = y_t - \frac{r}{1+r} \sum_{j=0}^{\infty} \frac{\mathbb{E}_t y_{t+j}}{(1+r)^j} \tag{58}$$

$$ca_t = y_t - \frac{r}{1+r} y_t - \frac{r}{1+r} \sum_{j=1}^{\infty} \frac{\mathbb{E}_t y_{t+j}}{(1+r)^j} \tag{59}$$

$$ca_t = \frac{1}{1+r} y_t - \frac{r}{1+r} \left(\frac{\mathbb{E}_t y_{t+1}}{(1+r)} + \frac{\mathbb{E}_t y_{t+2}}{(1+r)^2} + \dots \right) \tag{60}$$

add and substract $\frac{r}{1+r} \mathbb{E}_t y_{t+i}$ for all $i \in (0, \dots, \infty)$

$$\begin{aligned} ca_t =& \frac{1}{1+r} y_t - \frac{r}{1+r} \left(\frac{\mathbb{E}_t \Delta y_{t+1}}{(1+r)} + \frac{\mathbb{E}_t \Delta y_{t+2}}{(1+r)^2} + \dots \right) - \\ & \frac{r}{1+r} \left(\frac{\mathbb{E}_t y_t}{(1+r)} + \frac{\mathbb{E}_t y_{t+1}}{(1+r)^2} + \dots \right) \end{aligned} \tag{61}$$

$$\begin{aligned} ca_t =& \frac{1}{1+r} y_t - \frac{r}{1+r} \left(\frac{\mathbb{E}_t \Delta y_{t+1}}{(1+r)} + \frac{\mathbb{E}_t \Delta y_{t+2}}{(1+r)^2} + \dots \right) - \\ & \frac{r}{(1+r)} \frac{1}{(1+r)} \left(\mathbb{E}_t y_t + \frac{\mathbb{E}_t y_{t+1}}{(1+r)} + \frac{\mathbb{E}_t y_{t+2}}{(1+r)^2} + \dots \right) \end{aligned} \tag{62}$$

$$\begin{aligned} ca_t =& \frac{1}{1+r} y_t - \frac{r}{1+r} \left(\frac{\mathbb{E}_t \Delta y_{t+1}}{(1+r)} + \frac{\mathbb{E}_t \Delta y_{t+2}}{(1+r)^2} + \dots \right) - \\ & \frac{r}{(1+r)^2} \left(\sum_{j=0}^{\infty} \frac{\mathbb{E}_t y_{t+j}}{(1+r)^j} \right) \end{aligned} \tag{63}$$

Using equation 58

$$ca_t = \frac{1}{1+r}y_t - \frac{r}{1+r}\left(\frac{\mathbb{E}_t\Delta y_{t+1}}{(1+r)} + \frac{\mathbb{E}_t\Delta y_{t+2}}{(1+r)^2} + \dots\right) - \frac{r}{(1+r)^2}\left(\frac{1+r}{r}(y_t - ca_t)\right) \quad (64)$$

$$ca_t = \frac{1}{1+r}y_t - \frac{r}{1+r}\left(\frac{\mathbb{E}_t\Delta y_{t+1}}{(1+r)} + \frac{\mathbb{E}_t\Delta y_{t+2}}{(1+r)^2} + \dots\right) - \frac{1}{1+r}(y_t - ca_t) \quad (65)$$

$$ca_t = -\left(\frac{\mathbb{E}_t\Delta y_{t+1}}{(1+r)} + \frac{\mathbb{E}_t\Delta y_{t+2}}{(1+r)^2} + \dots\right) \quad (66)$$

$$ca_t = -\sum_{j=1}^{\infty}\frac{\mathbb{E}_t\Delta y_{t+j}}{(1+r)^j} \quad (67)$$

Use the stochastic endowment process

$$ca_t = -\sum_{j=1}^{\infty}\frac{\rho^j\Delta y_t}{(1+r)^j} \quad (68)$$

$$ca_t = -\frac{\rho}{1+r-\rho}\Delta y_t \quad (69)$$

Countercyclicality:

$$\frac{\mathrm{d}ca_t}{\mathrm{d}\varepsilon_t} = -\frac{\rho}{1+r-\rho} < 0 \quad (70)$$

Process of Δc_t:

$$ca_t \equiv tb_t - rd_{t-1} \quad (71)$$

$$ca_t = y_t - c_t - rd_{t-1} \quad (72)$$

$$ca_t - ca_{t-1} = \Delta y_t - \Delta c_t - r(d_{t-1} - d_{t-2}) \quad (73)$$

$$ca_t = \Delta y_t - \Delta c_t + (1+r)ca_{t-1} \quad (74)$$

Use equation 69 for ca_t

$$-\frac{\rho}{1+r-\rho}\Delta y_t = \Delta y_t - \Delta c_t - (1+r)\frac{\rho}{1+r-\rho}\Delta y_{t-1} \quad (75)$$

$$\Delta c_t = \frac{1+r}{1+r-\rho}(\Delta y_t - \rho \Delta y_{t-1}) \quad (76)$$

$$\Delta c_t = \frac{1+r}{1+r-\rho}\varepsilon_t \quad (77)$$

Consumption growth is white noise.

Relative volatility of consumption growth and endowment growth:

$$\Delta y_t = \rho \Delta y_{t-1} + \varepsilon_t \quad (78)$$

$$\Delta c_t = \frac{1+r}{1+r-\rho}\varepsilon_t \quad (79)$$

Note that ε_t is an i.i.d. zero mean shock process. Moreover, $\mathbb{E}_t(\varepsilon_t|\Delta y_{t-i}) = 0$ for $i > 0$ and $\mathbb{E}_t(\varepsilon_t^2|\Delta y_{t-i}) = \sigma_\varepsilon^2$ for all i.

Since $\varepsilon_t \sim$ i.i.d.

$$V(\Delta y) = \frac{1}{1-\rho^2}\sigma_\varepsilon^2 \quad (80)$$

Similarly for Δc_t

$$V(\Delta c) = \left(\frac{1+r}{1+r-\rho}\right)^2 \sigma_\varepsilon^2 \quad (81)$$

If $\rho = 0$ then $V(\Delta y) = V(\Delta c)$. If $\rho \to 1$ then $V(\Delta y) \to \infty$ and $V(\Delta y) > V(\Delta c)$. Finally, if $0 < \rho < 1$ then $V(\Delta c) > V(\Delta y)$.