Math Review

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International Monetary Economics

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1. The (representative) household problem.

   (a) The utility function, the felicity function, and the dynamic budget constraints (DBC) of this two period model ($t = 0$) are given by

$$U(c_t, c_{t+1}) = u(c_t) + \beta u(c_{t+1}) \tag{1}$$

$$u(c_i) = \frac{c_i^{1-\sigma}}{1-\sigma} \ \forall i \tag{2}$$

$$y_t = c_t + b_t \tag{3}$$

$$y_{t+1} + (1+r)b_t = c_{t+1} \tag{4}$$

   What important assumptions are implicit in the model setup?

   Solution:

   - *Utility: Time separable utility (no habit formation)*
   - *Utility: Discount factor $\beta = \frac{1}{1+\rho}$ with ρ: discount rate*
   - *Utility: Constant relative risk aversion σ*
   - *Utility: Constant intertemporal elasticity of substitution $\frac{1}{\sigma}$*
   - *DBC: Transversality condition: $b_{t+1} \leq 0$*
   - *DBC: No Ponzi condition: $b_{t+1} \geq 0$*
   - *DBC: Initial asset holding is zero: $b_{t-1} = 0$*

   (b) Derive the intertemporal budget constraint (IBC). Give an interpretation and discuss some of its implications.

   Solution:

$$y_t + \frac{y_{t+1}}{1+r} = c_t + \frac{c_{t+1}}{1+r} \tag{5}$$

   The net present value of the household's income equals the net present value of the household's consumption. In the absence of borrowing constraints, the allocation of consumption over time does not depend on the distribution of income over time.

(c) Is there a maximum amount of debt that the household can accumulate in period t?

Solution: *Yes, the natural borrowing limit. It is determined by the No Ponzi condition and reflected in the IBC. Debt in period* t *is maximized if nothing is consumed in period* $t+1$ *(i.e. in period* $t+1$*, the household only repays his debt).*

$$c_t = y_t + \frac{y_{t+1}}{1+r} \tag{6}$$

$$b_t = -\frac{y_{t+1}}{1+r} \tag{7}$$

A borrowing constraint usually refers to a tighter than natural borrowing limit (see previous exercise).

(d) Set up the household's optimization problem in the form of a Lagrangian. Show that you can do so in (at least) three different ways. Derive the Euler equation and given an intuitive interpretation.

Solution: Approach 1:

$$\mathcal{L} = U(c_t, c_{t+1}) + \lambda(y_t + \frac{y_{t+1}}{1+r} - c_t - \frac{c_{t+1}}{1+r}) \tag{8}$$

$$U'(c_t) \stackrel{!}{=} \lambda \tag{9}$$

$$U'(c_{t+1}) \stackrel{!}{=} \frac{\lambda}{1+r} \tag{10}$$

$$u'(c_t) = \lambda \tag{11}$$

$$\beta u'(c_{t+1}) = \frac{\lambda}{1+r} \tag{12}$$

$$u'(c_t) = \beta(1+r)u'(c_{t+1}) \tag{13}$$

Explicitly

$$c_t^{-\sigma} = \beta(1+r)c_{t+1}^{-\sigma} \tag{14}$$

$$\left(\frac{c_{t+1}}{c_t}\right)^{\sigma} = \beta(1+r) \tag{15}$$

$$\frac{c_{t+1}}{c_t} = (\beta(1+r))^{\frac{1}{\sigma}} \tag{16}$$

Approach 2:

$$\mathcal{L} = U(c_t, c_{t+1}) + \lambda_t(y_t - c_t - b_t) + \beta\lambda_{t+1}(y_{t+1} + (1+r)b_t - c_{t+1}) \tag{17}$$

$$U'(c_t) \stackrel{!}{=} \lambda_t \tag{18}$$

$$U'(c_{t+1}) \stackrel{!}{=} \beta\lambda_{t+1} \tag{19}$$

$$\lambda_t \stackrel{!}{=} \beta(1+r)\lambda_{t+1} \tag{20}$$

$$u'(c_t) = \lambda_t \tag{21}$$

$$u'(c_{t+1}) = \lambda_{t+1} \tag{22}$$

$$\lambda_t = \beta(1+r)\lambda_{t+1} \tag{23}$$

$$u'(c_t) = \beta(1+r)u'(c_{t+1}) \tag{24}$$

Approach 3:

$$\mathcal{L} = U(c_t, c_{t+1}) + \Lambda_t(y_t - c_t - b_t) + \Lambda_{t+1}(y_{t+1} + (1+r)b_t - c_{t+1}) \tag{25}$$

$$U'(c_t) \stackrel{!}{=} \Lambda_t \tag{26}$$

$$U'(c_{t+1}) \stackrel{!}{=} \Lambda_{t+1} \tag{27}$$

$$\Lambda_t \stackrel{!}{=} (1+r)\Lambda_{t+1} \tag{28}$$

$$u'(c_t) = \Lambda_t \tag{29}$$

$$\beta u'(c_{t+1}) = \Lambda_{t+1} \tag{30}$$

$$\Lambda_t = (1+r)\Lambda_{t+1} \tag{31}$$

$$u'(c_t) = \beta(1+r)u'(c_{t+1}) \tag{32}$$

Formally: *The Euler equation is an optimality condition which equates the current marginal utility of consumption (what you get in terms of utility if you consumed an additional unit in the current period) and the discounted marginal utility of consuming an additional* $(1+r)$ *unit tomorrow.*

Intuition: *You can either consume an additional unit today (and get* $u'(c_t)$*) or save it. If you save it, you have an additional* $(1+r)$ *unit to consume tomorrow (which gives you* $(1+r)u'(c_{t+1})$*). Because you are impatient, getting* $(1+r)u'(c_{t+1})$ *in the next period is only worth* $\beta(1+r)u'(c_{t+1})$ *in the current period.*

If the marginal utilities of the two options (consuming one additional unit today and saving one additional unit today) were not the same, the agents could increase their utility by re-allocating consumption (e.g. consume more today and less tomorrow). An optimal allocation of consumption requires the marginal utilities of the two options to be equal.

2. Consumption growth and the intertemproal elasticity of substitution.

   (a) Proof that $\ln\left(\frac{c_{t+1}}{c_t}\right)$ is (approximately) equal to the growth rate of consumption.

   Solution: *A first order Taylor approximation of a function* $f(x)$ *around* $x = a$ *is*

$$f(x) \approx f(a) + \frac{1}{1!}f'(a)(x-a) \tag{33}$$

   Use this result to approximate the right hand side of the following *equation around a zero growth rate of consumption.*

$$\frac{c_{t+1} - c_t}{c_t} \overset{?}{\approx} ln\left(\frac{c_{t+1}}{c_t}\right) \tag{34}$$

$$\frac{c_{t+1} - c_t}{c_t} \overset{?}{=} ln(1) + \frac{1}{1!}\frac{1}{1}\left(\frac{c_{t+1}}{c_t} - 1\right) \tag{35}$$

$$\frac{c_{t+1} - c_t}{c_t} \overset{?}{=} \frac{c_{t+1}}{c_t} - 1 \tag{36}$$

$$\frac{c_{t+1} - c_t}{c_t} \overset{!}{=} \frac{c_{t+1} - c_t}{c_t} \tag{37}$$

(b) Re-express the Euler equation in terms of the discount rate (rather than the discount factor). How does σ affect the growth rate of consumption if $r > \rho$?

Solution:

$$\frac{c_{t+1}}{c_t} = \left(\frac{1+r}{1+\rho}\right)^{\frac{1}{\sigma}} \tag{38}$$

$$ln\left(\frac{c_{t+1}}{c_t}\right) = \frac{1}{\sigma}\left(ln(1+r) - ln(1+\rho)\right) \tag{39}$$

$$ln\left(\frac{c_{t+1}}{c_t}\right) \approx \frac{1}{\sigma}\left(r - \rho\right) \tag{40}$$

- $\rho = r \leftrightarrow \beta(1+r) = 1$: *constant consumption*
- $\rho < r \leftrightarrow \beta(1+r) > 1$: *increasing consumption*
- $\rho > r \leftrightarrow \beta(1+r) < 1$: *decreasing consumption*

$$\frac{dln\left(\frac{c_{t+1}}{c_t}\right)}{d\sigma} = -\frac{1}{\sigma^2}(r - \rho) < 0 \tag{41}$$

With $r > \rho$, an increase in σ decreases consumption growth. Consumption is still growing (since $r > \rho$) but it does so at a slower rate. More formally, an increase in σ is akin to a decrease in the intertemporal elasticity of substitution, i.e. it is akin to a decrease in the willingness to substitute consumption across periods. The *lower this willingness, the smoother the consumption path. For $\sigma \to \infty$, consumption is constant even if $r > \rho$.*

(c) What is the (relative) price of the consumption good in period $t+1$ (p_{t+1}) if you set $p_t = 1$?

Solution: *Follows directly from the intertemporal budget constraint:* $p_2 = \frac{1}{1+r}$.

(d) Proof formally that σ is the inverse intertemporal elasticity of subsitution. Use $p_t = 1$.

Solution:

$$\varepsilon_{t,t+1} = -\frac{\partial \frac{c_t}{c_{t+1}}}{\partial \frac{p_t}{p_{t+1}}} \frac{\frac{p_t}{p_{t+1}}}{\frac{c_t}{c_{t+1}}} \tag{42}$$

Compute the first term

$$\frac{c_t}{c_{t+1}} = \left(\frac{1+r}{1+\rho}\right)^{-\frac{1}{\sigma}} \tag{43}$$

$$\frac{\partial \frac{c_t}{c_{t+1}}}{\partial \frac{p_t}{p_{t+1}}} = -\frac{1}{\sigma} \left(\frac{1+r}{1+\rho}\right)^{-\frac{1}{\sigma}-1} \left(\frac{1}{1+\rho}\right) \tag{44}$$

Plug the above equation into the equation for the (intertemporal) elasticity of substitition

$$\varepsilon_{t,t+1} = \frac{1}{\sigma} \left(\frac{1+r}{1+\rho}\right)^{-\frac{1}{\sigma}-1} \left(\frac{1}{1+\rho}\right) \frac{\frac{p_t}{p_{t+1}}}{\frac{c_t}{c_{t+1}}} \tag{45}$$

$$\varepsilon_{t,t+1} = \frac{1}{\sigma} \left(\frac{1+r}{1+\rho}\right)^{-\frac{1}{\sigma}-1} \left(\frac{1}{1+\rho}\right) \frac{1+r}{\left(\frac{1+r}{1+\rho}\right)^{-\frac{1}{\sigma}}} \tag{46}$$

$$\varepsilon_{t,t+1} = \frac{1}{\sigma}(1+r)^{-\frac{1}{\sigma}-1+1+\frac{1}{\sigma}}(1+\rho)^{\frac{1}{\sigma}+1-1-\frac{1}{\sigma}} \tag{47}$$

$$\varepsilon_{t,t+1} = \frac{1}{\sigma} \tag{48}$$