Problem Set 4

Department of Economics at the University of Bern

International Monetary Economics

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1. Consider a small open Arrow-Debreu endowment economy with two countries $h = \{1,2\}$, two periods $t = \{1,2\}$, and two states $s = \{1,2\}$ in period 2. The countries differ only in endowments $w_t^h(s)$. That is, they maximize an analogous (log-) utility function over consumption $x_t^h(s)$ with identical discount factors and equal believes about the probabilities of each state.

$$U^h = log(x_1^h) + \beta \mathbb{E}_1 log(x_2^h) \quad \forall h \tag{1}$$

   (a) Find the intertemporal budget constraint (IBC) of the representative agent in country h assuming zero initial assets. Use $q(s)$ for the price of a (state-contigent) Arrow-Debreu security b_s^h that pays one unit in state s and zero otherwise.

   (b) Show that the first order conditions (FOC) of the agents can be written as

$$\frac{q(2)}{q(1)} = \frac{\pi(2)}{\pi(1)} \frac{x_2^h(1)}{x_2^h(2)} \tag{2}$$

$$x_2^h(s) = \frac{\pi(s)}{q(s)} \beta x_1^h \tag{3}$$

   (c) Use equation 3 in the IBC to get

$$x_1^h = \frac{w_1^h + q(1)w_2^h(1) + q(2)w_2^h(2)}{1+\beta} \tag{4}$$

Solution:

$$w_1^h - x_1^h - q(1)\left[\frac{\pi(1)}{q(1)}\beta x_1^h - w_2^h(1)\right] - q(2)\left[\frac{\pi(2)}{q(2)}\beta x_1^h - w_2^h(2)\right] = 0 \tag{5}$$

$$x_1^h + \pi(1)\beta x_1^h + \pi(2)\beta x_1^h = w_1^h + q(1)w_2^h(1) + q(2)w_2^h(2) \tag{6}$$

$$x_1^h + \beta x_1^h = w_1^h + q(1)w_2^h(1) + q(2)w_2^h(2) \tag{7}$$

$$(1+\beta)x_1^h = w_1^h - q(1)w_2^h(1) - q(2)w_2^h(2) \tag{8}$$

$$x_1^h = \frac{w_1^h + q(1)w_2^h(1) + q(2)w_2^h(2)}{1+\beta} \tag{9}$$

(d) Combine the two market clearing conditions in period 2 to obtain an expression for the relative price of the two Arrow-Debreu securities in terms of the two states' probabilities and the two states' (aggregated) endowments. Why does the relative price of the second security $q(2)$ rise in the aggregated endowment of state 1?

Solution: *The two market clearing conditions are given by*

$$w_2^a(1) = x_2^1(1) + x_2^2(1) \tag{10}$$

$$w_2^a(2) = x_2^1(2) + x_2^2(2) \tag{11}$$

with $w_t^a(s) \equiv w_t^1(s) + w_t^2(s)$ *being the aggregated endowment in time* t*, state* s*. Using the FOC (equation 2) in the market clearing condition under* $s = 1$ *yields*

$$\frac{q(2)}{q(1)}\frac{\pi(1)}{\pi(2)}\left[x_2^1(2) + x_2^2(2)\right] = w_2^a(1) \tag{12}$$

Next, use the market clearing condition under $s = 2$

$$\frac{q(2)}{q(1)} \frac{\pi(1)}{\pi(2)} w_2^a(2) = w_2^a(1) \tag{13}$$

$$\frac{q(2)}{q(1)} = \frac{\pi(2)}{\pi(1)} \frac{w_2^a(1)}{w_2^a(2)} \tag{14}$$

The relative price of $q(2)$ *rises in the aggregated endowment of state 1 because of the agents' risk aversion. A lower (relative) endowment in state 2 induces a greater willingness to buy a security which pays in state 2. Since market must clear in equilibrium, the relative price of the security paying in state 2 must rise.*[1]

(e) Find the (absolute) equilibrium prices of the two Arrow-Debreu securities. Discuss the determinants of the equilibrium prices.

Solution: *By definition, equilibrium prices clear (all) markets. Hence, we impose market clearing in period 1*

$$w_1^a = x_1^1 + x_1^2 \tag{15}$$

Next, make use of the result from c. (equation 4)

$$w_1^a = \frac{w_1^1 + q(1)w_2^1(1) + q(2)w_2^1(2)}{1+\beta} + \frac{w_1^2 + q(1)w_2^2(1) + q(2)w_2^2(2)}{1+\beta} \tag{16}$$

$$(1+\beta)w_1^a = w_1^a + q(1)w_2^1(1) + q(2)w_2^1(2) + q(1)w_2^2(1) + q(2)w_2^2(2) \tag{17}$$

$$\beta w_1^a = q(1)\left[w_2^1(1) + w_2^2(1)\right] + q(2)\left[w_2^1(2) + w_2^2(2)\right] \tag{18}$$

$$\beta w_1^a = q(1)w_2^a(1) + q(2)w_2^a(2) \tag{19}$$

Replace $q(1)$ *with the result from d. (equation 14) to get the (ab-*

[1] cf. chapter 2, exercise 5 in Obstfeld, Rogoff, et al. (1996) for an example with risk neutral agents with respect to period 2 consumption. In this case, the (relative) price of the Arrow-Debreu securities is solely determined by the two states' relative probabilites.

solute) equilibrium price of $q(2)$

$$\beta w_1^a = q(2)\frac{\pi(1)}{\pi(2)}\frac{w_2^a(2)}{w_2^a(1)}w_2^a(1) + q(2)w_2^a(2) \tag{20}$$

$$\beta w_1^a = q(2)w_2^a(2)\left(1 + \frac{\pi(1)}{\pi(2)}\right) \tag{21}$$

$$\beta w_1^a = q(2)\frac{w_2^a(2)}{\pi(2)} \tag{22}$$

$$q(2) = \pi(2)\beta\frac{w_1^a}{w_2^a(2)} \tag{23}$$

Using, again, the result from c. (equation 14) we get the (absolute) equilibrium price of $q(1)$*:*

$$q(1) = \pi(1)\beta\frac{w_1^a}{w_2^a(1)} \tag{24}$$

The (absolute) prices of the Arrow-Debreu securities are determined by three factors. First, time preferences. The reason is that the securities are not only used to insure against the endowment risk but also to transfer resources across periods. The higher the time preference (the lower β*), the cheaper the equilibrium price of (both) Arrow-Debreu securities. Second, believes about the likelihood of the states. The higher the probability associated to a state, the more expensive is the security paying in that state. Third, aggregated endowment. The higher the current endowment relative to future endowment in state* j*, the higher the price of the security paying in state* j*.*[2] *The reason is that agents wish to smooth consumption not only across states but also across time. Hence, they wish to bring resources into the future if their current endowment is comparatively high. Since markets must clear in equilibrium, it is more expensive to bring resources into the period in which your endowment is lower.*[3]

[2] cf. exercise 1, d. for a discussion of the effect of differences in the endowment across states on the relative price of the two Arrow-Debreu securities.

[3] **Remark 1:** We did not normalize one of the two securities to 1 because the (implicit)

2. Consider a small open endowment economy with two periods $t = \{1, 2\}$ and two states $s = \{1, 2\}$ in period 2. The representative agent can use two state-contingent Arrow-Debreu securites b_s and a non-contigent bond b_3 as means of transfering resources across time and states. The Arrow-Debreu securities are traded at prices $q(s)$ and pay one unit in state s and zero otherwise. The bond pays $1 + r$ in both states. Its price $q(3)$ is normalized to 1. Initial asset holdings are zero.

   (a) Derive the dynamic budget constraints (DBC) of the agent.

   (b) Use an appropriate (log-) utility function over consumption $x_t(s)$ to proof that $q(1)+q(2) = \frac{1}{1+r}$ and give an economic interpretation of your result.[4]

   (c) Reconsider the results from exercise 1, e, $q(1) = \pi(1)\beta\frac{w_1^a}{w_2^a(1)}$ and $q(2) = \pi(2)\beta\frac{w_1^a}{w_2^a(2)}$, and the result from exercise 2, b, $\frac{1}{1+r} = q(1) + q(2)$. What determines the value of $\beta(1+r)$? How is this different from problem set 1, exercise 2, c? You may want to resort to the definition of convexity (a function $u(x)$ is strictly convex if $\mathbb{E}(u(x)) > u(\mathbb{E}(x))$ and use $v(w_2^a) = \frac{w_1^a}{w_2^a}$.

   Solution: *Let us first combine the results from exercise 1,e and the result from exercise 2, b.*

$$\frac{1}{1+r} = \pi(1)\beta\frac{w_1^a}{w_2^a(1)} + \pi(2)\beta\frac{w_1^a}{w_2^a(2)} \tag{25}$$

$$\frac{1}{\beta(1+r)} = \pi(1)\frac{w_1^a}{w_2^a(1)} + \pi(2)\frac{w_1^a}{w_2^a(2)} \tag{26}$$

$$\frac{1}{\beta(1+r)} = \mathbb{E}_1\left(\frac{w_1^a}{w_2^a}\right) \tag{27}$$

numeraire is the price of period 1 consumption. **Remark 2:** The analysis could have been much richer if we had allowed for differences in the discount factors and/or in the believes about the probabilities of each state.

[4] cf. ime_asset_20170313, p. 5.

In problem set 1, exercise 2, c, $\beta(1+r)$ was determined by the deterministic relative aggregated endowment over the two periods. In particular,

$$\frac{1}{\beta(1+r)} = \frac{Y_1 + Y_1^\star}{Y_2 + Y_2^\star} \tag{28}$$

Here, the relative aggregated endowment in the two periods is no longer know for sure as of period 1. Hence, $\beta(1+r)$ is determined by the expected relative endowment over the two periods. Because $v(w_2^a)$ is strictly convex in w_2^a, we have that $\mathbb{E}_1(v(w_2^a)) > v(\mathbb{E}_1(w_2^a))$. Suppose that the average of the two potential endowments in $s=1$ and $s=2$ is equal to the deterministic aggregated endowment as in problem set 1, i.e. $Y_2 + Y_2^\star$. Then we find that the equilibrium interest rate is lower if agents have to face a risky endowment in period 2. The reason is that it is valueable to risk averse agents to insure away the endowment risk. Consequently, they are willing to accept lower returns on their securities.

References

Obstfeld, M., K. S. Rogoff, et al. (1996): *Foundations of international macroeconomics*, vol. 30. MIT press Cambridge, MA.