Problem Set 5

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International Monetary Economics

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1. Comparing optimal consumption with complete and incomplete markets (Obstfeld, Rogoff, et al. (1996), chapter 5, exercise 3). Consider a two-period small open endowment economy facing the world interest rate r for riskless loans. Date 1 output is Y_1. There are $\mathcal{S}$ states of nature on date 2 that differ according to the associated output realizations $Y_2(s)$ and have probabiliteies $\pi(s)$ of occuring. The representative domestic consumer maximizes the expected lifetime utility function

$$U_1 = C_1 - \frac{\alpha_0}{2}(C_1)^2 + \frac{1}{1+r}\mathbb{E}_1\left\{C_2(s) - \frac{\alpha_0}{2}(C_2(s))^2\right\} \tag{1}$$

in which period utility is quadratic and $\alpha_0 > 0$. The relevant dynamic budget constraints (DBC) when markets are incomplete can be written as

$$B_2 = (1+r)B_1 + Y_1 - C_1 \tag{2}$$

$$C_2(s) = (1+r)B_2 + Y_2(s) \tag{3}$$

for $s = 1, 2, \ldots, \mathcal{S}$ and B_1 given. The last constraint is equivalent to the $\mathcal{S}$ intertemporal budget constraints (IBC): for all states s,

$$C_1 + \frac{C_2(s)}{1+r} = (1+r)B_1 + Y_1 + \frac{Y_2(s)}{1+r} \tag{4}$$

You may assume that all output levels are small enough that the marginal utility of consumption $1 - \alpha_0 C$ is safely positive.

(a) In the model it is assumed that the discount factor β is equal to the inverse of the gross nominal interest rate. What is the (implicit) assumption about the expected aggregated endowment over time?

Solution: *Suppose that the aggregated endowment in period* 1 *is below the expected aggregated endowment in period* 2*. Then, in the aggregate, agents would like to transfer resources from the* 2^{nd} *to the* 1^{st} *period. Consequently, the market clearing interest rate would lie above* $\frac{1}{\beta}$*. The argument applies vice versa too, hence the aggregated endowment in period* 1 *must equal the expected aggregated endowment in period* 2 *if* $\beta(1+r)=1$.[1]

(b) Start by temporarily ignoring the nonnegativity constraints $C_2(s) \geq 0$ on date 2 consumption. Compute optimal date 1 consumption C_1^{dis}. What are the implied values of $C_2^{dis}(s)$?

(c) Now let's worry about the nonnegativity constraint on C_2. Renumber the date 2 states of nature (if necessary) so that $Y_2(1) = \min Y_2(s)$. Show that if

$$(1+r)B_1 + Y_1 + \frac{2+r}{1+r}Y_2(1) \geq \mathbb{E}_1(Y_2(s)) \tag{5}$$

then the C_1 computed in part b. (for the two period case) is still valid. What is the intuition? Suppose the preceding inequality does not hold. Show that the optimal date 1 consumption is lower (a precautionary saving effect) and equals

$$C_1 = (1+r)B_1 + Y_1 + \frac{Y_2(1)}{1+r} \tag{6}$$

Explain the preceding answer under the simplifying assumption $Y_1 = \mathbb{E}_1(Y_2(s))$. Does the bond Euler equation hold in this case? Hint: Apply the Kuhn-Tucker theorem.

Solution: *The consumption formula of part b. will generally be valid if the nonnegativity constraint on consumption never binds, that is, if, even when output hits its minimal date 2 value (in state* $s=1$*),* $C_2 \geq 0$*. From the DBC of period 2, this last inequality*

[1] cf. problem set 4, exercise 2, c.

will hold if and only if

$$(1+r)B_2 + Y_2(1) \geq 0 \tag{7}$$

From the DBC of period 1 where C_1^{dis} *is given from exercise b.*

$$(1+r)\left\{(1+r)B_1 + Y_1 - C_1^{dis}\right\} + Y_2(1) \geq 0 \tag{8}$$

$$(1+r)\left\{(1+r)B_1 + Y_1 - \frac{1+r}{2+r}\left[(1+r)B_1 + Y_1 + \frac{\mathbb{E}_1(Y_2(s))}{1+r}\right]\right\} + Y_2(1) \geq 0 \tag{9}$$

$$(1+r)B_1 + Y_1 - \frac{1+r}{2+r}\left[(1+r)B_1 + Y_1 + \frac{\mathbb{E}_1(Y_2(s))}{1+r}\right] \geq -\frac{Y_2(1)}{1+r} \tag{10}$$

$$(2+r)B_1 + \frac{2+r}{1+r}Y_1 - (1+r)B_1 - Y_1 - \frac{\mathbb{E}_1(Y_2(s))}{1+r} \geq -\frac{2+r}{1+r}\frac{Y_2(1)}{1+r} \tag{11}$$

$$B_1 + \frac{1}{1+r}Y_1 - \frac{\mathbb{E}_1(Y_2(s))}{1+r} \geq -\frac{2+r}{1+r}\frac{Y_2(1)}{1+r} \tag{12}$$

$$(1+r)B_1 + Y_1 + \frac{2+r}{1+r}Y_2(1) \geq \mathbb{E}_1(Y_2(s)) \tag{13}$$

If this inequality does not hold, then the nonnegativity constraint on C_2 *binds in at least one state of nature on date 2, so we cannot ignore the associated Kuhn-Tucker multiplier.*[2] *In that case, the Kuhn-Tucker theorem predicts that date 1 consumption must make* $C_2(1) = 0$ *(in state 1 of date 2 when output is minimal). From using* $C_2(1) = 0$ *in the IBC*

$$C_2(1) = (1+r)\left\{(1+r)B_1 + Y_1 - C_1\right\} + Y_2(1) = 0 \tag{14}$$

we see that

$$C_1 = (1+r)B_1 + Y_1 + \frac{Y_2(1)}{1+r} \tag{15}$$

Interpretation: If the nonnegativity constraint never binds, the result from b. is still valid (logically, the bond Euler equation derived in b. must also hold). The reason is that the marginal cost associated to the nonnegativity constraint is zero because the constraint

[2] Assume that endowments are weakly positive in all states. Moreover, recall the simplifying assumption $Y_1 = \mathbb{E}_1(Y_2(s))$. It follows that the inequality can only be violated if $B_1 < 0$.

is not relevant for the agents. Conversely, if the nonnegativity constraint may (!) bind, the marginal cost associated to the constraint is positive. The Euler equation derived in b. would not hold.

(d) Use the results from b. and c. to proof that consumption in period 1 is indeed smaller ifthe nonnegativity constraint on $C_2(s)$ may bind. Assume that $Y_1 = \mathbb{E}_1(Y_2(s))$.

Solution: *Formally, we want to proof that*

$$C_1^{dis} > C_1 \tag{16}$$

if $(1+r)B_1 + \frac{2+r}{1+r}Y_2(1) < 0$, *i.e. if the nonnegativity constraint may bind. We proof this by contracdiction. Suppose that the nonnegativity constraint never binds and* $C_1^{dis} > C_1$. *Using the results from b. and c. yields*

$$\frac{1+r}{2+r}\mathbb{E}_1\Big\{(1+r)B_1 + Y_1 + \frac{Y_2(j)}{1+r}\Big\} > (1+r)B_1 + Y_1 + \frac{Y_2(1)}{1+r} \tag{17}$$

$$(1+r)^2 B_1 + (1+r)Y_1 + \mathbb{E}_1(Y_2(s)) > (2+r)(1+r)B_1 + (2+r)Y_1 + \frac{2+r}{1+r}Y_2(1) \tag{18}$$

$$(1 + r^2 + 2r - 2 - 2r - r - r^2)B_1 > \frac{2+r}{1+r}Y_2(1) \tag{19}$$

$$0 > (1+r)B_1 + \frac{2+r}{1+r}Y_2(1) \tag{20}$$

a contradiction (the nonnegativity constraint binds).

(e) Now assume the consumer faces complete global asset markets with $p(s)$, the state s Arrow-Debreu security price, equal to $\pi(s)$. Find the optimal values of C_1 and $C_2(s)$ now. Why can nonnegativity constraints be disregarded in the complete markets case?

2. Consider a simple monetary model of the nominal exchange rate. Let M_t denote nominal money supply, P_t, the price level, i_t the nominal net interest rates, Y_t real output, and ε_t the nominal exchange rate

(in domestic currency units per foreign currency units). Small letters indicate the logs of the corresponding level-variables and foreign variables are distinguished by asteriks. Furthermore, use $\mathbf{i}_t \equiv log(1 + i_t)$ and $e_t = log(\varepsilon_t)$. Finally, assume $\eta > 0$ and $\phi > 0$. The money demand equation, the purchasing power parity (PPP) and the uncovered interest rate parity (UIP) are given by

$$\frac{M_t}{P_t} = \frac{Y_t^{\phi}}{(1 + i_{t+1})^{\eta}} \tag{21}$$

$$P_t = \varepsilon_t P_T^{\star} \tag{22}$$

$$(1 + i_t) = (1 + i_t^{\star})\mathbb{E}_t \left(\frac{\varepsilon_{t+1}}{\varepsilon_t} \right) \tag{23}$$

(a) Suppose that the exchange rate in time $t + 1$ is known for sure as of period t. Express the model in logs.

Solution:

$$m_t - p_t = -\eta \mathbf{i}_t + \phi y_t \tag{24}$$

$$p_t = e_t + p_t^{\star} \tag{25}$$

$$\mathbf{i}_t = \mathbf{i}_t^{\star} + e_{t+1} - e_t \tag{26}$$

(b) Now, suppose that agents have to make a forecast of the exchange rate in time $t + 1$. The model in logs is then approximated by

$$m_t - p_t = -\eta \mathbf{i}_t + \phi y_t \tag{27}$$

$$p_t = e_t + p_t^{\star} \tag{28}$$

$$\mathbf{i}_t = \mathbf{i}_t^{\star} + \mathbb{E}_t e_{t+1} - e_t \tag{29}$$

Why is this model representation only an approximation of the original model? You may want to resort to the definition of concavity (a function $u(x)$ is strictly concave if $\mathbb{E}(u(x)) < u(\mathbb{E}(x))$ and use $v(\varepsilon_{t+1}) = log\left(\frac{\varepsilon_{t+1}}{\varepsilon_t}\right)$).

(c) Let m_t, y_t, and all foreign variables be exogenous variables. Find the solution for e_t. What is the (qualitative) response of e_t to an increase in the nominal money supply m_s?

(d) What is the (quantitative) response of e_t and $\mathbf{i}_t$ to an increase in the contemporaneous nominal money supply m_t?

Solution: *First, take the total differential with respect to e_t and m_t in the result of the previous exercise.*

$$de_t = \frac{1}{1+\eta} dm_t \tag{30}$$

The nominal exchange rate rises by $\frac{1}{1+\eta} dm_t$ in m_t. Second, take the total differential on the PPP equation with respect to p_t and e_t.

$$dp_t = de_t \tag{31}$$

$$dp_t = \frac{1}{1+\eta} dm_t \tag{32}$$

Third, take the total differential on the money demand equation with respect to m_t, p_t and $\mathbf{i}_t$.

$$dm_t - dp_t = -\eta d\mathbf{i}_t \tag{33}$$

$$dm_t - \frac{1}{1+\eta} dm_t = -\eta d\mathbf{i}_t \tag{34}$$

$$d\mathbf{i}_t = -\frac{1}{1+\eta} dm_t < 0 \tag{35}$$

The nominal interest rate falls by $\frac{1}{1+\eta} dm_t$ in m_t.

(e) What is the (qualitative) response of e_t to a decrease in the nominal interest rate? How do you explain that an interest rate shock induces $corr(e_t, \mathbf{i}_t) > 0$ while a nominal money supply shock implies $corr(e_t, \mathbf{i}_t) < 0$?

Solution: *The interest rate shock is conceptually different from the nominal money supply shock because it does not induce a change in the (exogenous) nominal money supply. Because nominal money supply is exogenous (and constant when we consider an interest rate shock), the money demand equation requires that $corr(p_t, \mathbf{i}_t) > 0$. In other words, a lower nominal interest rate translates into a lower price level. In turn, an decrease in p_t induces an decrease in e_t (an appreciation) via the PPP equation.*
In contrast, the nominal money supply shock affects the (endogenous) interest rate. In this sense, only a nominal money supply shock is in effect expansionary.

(f) Find a way to show that the nominal exchange rate depreciates more when the *change* in the nominal money supply exhibits positive persistence. Assume that $\eta \mathbf{i}_s^\star - p_s^\star - \phi y_s = 0$.[3]

Solution: *We can model positive persistence in the growth of nominal money supply as an AR(1) process with zero mean, $\rho \in (0,1)$, and constant variance. Formally,*

$$m_t - m_{t-1} = \rho(m_{t-1} - m_{t-2}) + \zeta_t^m \tag{36}$$

with $\mathbb{E}_t \zeta_{t+i}^m = 0 \ \forall i$ and $V(\zeta_{t+i}^m) = \sigma_m^2 \ \forall i$. The expected change in the nominal money supply in period $t+S$ is then given by

$$\mathbb{E}_t \left(m_{t+S} - m_{t+S-1} \right) = p^S \left(m_t - m_{t-1} \right) \tag{37}$$

Lead the solution for e_t (equation 34) by one period, take expecta-

[3]cf. Obstfeld, Rogoff, et al. (1996, p. 526-530)

tions as of period t and substract the solution for e_t to get

$$\mathbb{E}_t e_{t+1} - e_t = \frac{1}{1+\eta} \sum_{s=t}^{\infty} \left(\frac{\eta}{1+\eta} \right)^{s-t} \mathbb{E}_t \left(m_{s+1} - m_s \right) \tag{38}$$

Combine the last two expressions

$$\mathbb{E}_t e_{t+1} - e_t = \frac{1}{1+\eta} \sum_{s=t}^{\infty} \left(\frac{\eta\rho}{1+\eta} \right)^{s-t} \rho \left(m_t - m_{t-1} \right) \tag{39}$$

$$\mathbb{E}_t e_{t+1} - e_t = \frac{\rho}{1+\eta-\eta\rho} \left(m_t - m_{t-1} \right) \tag{40}$$

$$\mathbb{E}_t e_{t+1} - e_t = \frac{\rho}{1+\eta(1-\rho)} \left(m_t - m_{t-1} \right) \tag{41}$$

Substitute this expression into the original model equation (equation 31) an solve for e_t

$$m_t - e_t = -\frac{\eta\rho}{1+\eta(1-\rho)} \left(m_t - m_{t-1} \right) \tag{42}$$

$$e_t = m_t + \frac{\eta\rho}{1+\eta(1-\rho)} \left(m_t - m_{t-1} \right) \tag{43}$$

It follows that

$$de_t = dm_t + \frac{\eta\rho}{1+\eta(1-\rho)} dm_t \tag{44}$$

$$de_t = \frac{1+\eta}{1+\eta(1-\rho)} dm_t > \frac{1}{1+\eta} dm_t \tag{45}$$

A change in the current nominal money supply has a greater effect on the current nominal exchange rate if the change in the nominal money supply exhibits positive persistence. This is so because a shock to the current nominal money supply does not only increase current nominal money supply but also the expected future nominal money supply.

References

Obstfeld, M., K. S. Rogoff, et al. (1996): *Foundations of international macroeconomics*, vol. 30. MIT press Cambridge, MA.