Problem Set 5

Department of Economics at the University of Bern

International Monetary Economics

Please bring errors to the author's attention. Contact:
fabio.canetg@vwi.unibe.ch

Author:	Fabio Canetg
Address:	Schanzeneckstrasse 1
	3001 Bern
	Switzerland
E-Mail:	fabio.canetg@vwi.unibe.ch
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1. Comparing optimal consumption with complete and incomplete markets (Obstfeld, Rogoff, et al. (1996), chapter 5, exercise 3). Consider a two-period small open endowment economy facing the world interest rate r for riskless loans. Date 1 output is Y_1. There are $\mathcal{S}$ states of nature on date 2 that differ according to the associated output realizations $Y_2(s)$ and have probabiliteies $\pi(s)$ of occuring. The representative domestic consumer maximizes the expected lifetime utility function

$$U_1 = C_1 - \frac{\alpha_0}{2}(C_1)^2 + \frac{1}{1+r}\mathbb{E}_1\left\{C_2(s) - \frac{\alpha_0}{2}(C_2(s))^2\right\} \tag{1}$$

in which period utility is quadratic and $\alpha_0 > 0$. The relevant dynamic budget constraints (DBC) when markets are incomplete can be written as

$$B_2 = (1+r)B_1 + Y_1 - C_1 \tag{2}$$
$$C_2(s) = (1+r)B_2 + Y_2(s) \tag{3}$$

for $s = 1, 2, \ldots, \mathcal{S}$ and B_1 given. The last constraint is equivalent to the $\mathcal{S}$ intertemporal budget constraints (IBC): for all states s,

$$C_1 + \frac{C_2(s)}{1+r} = (1+r)B_1 + Y_1 + \frac{Y_2(s)}{1+r} \tag{4}$$

You may assume that all output levels are small enough that the marginal utility of consumption $1 - \alpha_0 C$ is safely positive.

(a) In the model it is assumed that the discount factor β is equal to the inverse of the gross nominal interest rate. What is the (implicit) assumption about the expected aggregated endowment over time?

Solution: *Suppose that the aggregated endowment in period* 1 *is below the expected aggregated endowment in period* 2. *Then, in the aggregate, agents would like to transfer resources from the* 2^{nd} *to the* 1^{st} *period. Consequently, the market clearing interest rate would lie above* $\frac{1}{\beta}$.[1] *The argument applies vice versa too, hence the aggregated endowment in period* 1 *must equal the expected aggregated endowment in period* 2 *if* $\beta(1+r) = 1$.[2]

(b) Start by temporarily ignoring the nonnegativity constraints $C_2(s) \geq 0$ on date 2 consumption. Compute optimal date 1 consumption C_1^{dis}. What are the implied values of $C_2^{dis}(s)$?

(c) Now let's worry about the nonnegativity constraint on C_2. Renumber the date 2 states of nature (if necessary) so that $Y_2(1) = \min Y_2(s)$. Show that if

$$(1+r)B_1 + Y_1 + \frac{2+r}{1+r}Y_2(1) \geq \mathbb{E}_1(Y_2(s)) \qquad (5)$$

then the C_1 computed in part b. (for the two period case) is still valid. What is the intuition? Suppose the preceding inequality does not hold. Show that the optimal date 1 consumption is lower (a precautionary saving effect) and equals

$$C_1 = (1+r)B_1 + Y_1 + \frac{Y_2(1)}{1+r} \qquad (6)$$

Explain the preceding answer under the simplifying assumption $Y_1 = \mathbb{E}_1(Y_2(s))$. Does the bond Euler equation hold in this case? Hint: Apply the Kuhn-Tucker theorem.

Solution: *The consumption formula of part b. will generally be valid if the nonnegativity constraint on consumption never binds, that is, if, even when output hits its minimal date 2 value (in state*

[1] The argument would have been more involved if we had not assumed quadratic utility, cf. exercise 1.b. and problem set 4, exercise 2, c.

[2] cf. problem set 4, exercise 2, c.

s = 1), $C_2 \geq 0$. From the DBC of period 2, this last inequality will hold if and only if

$$(1+r)B_2 + Y_2(1) \geq 0 \tag{7}$$

From the DBC of period 1 where C_1^{dis} is given from exercise b.

$$(1+r)\Big\{(1+r)B_1 + Y_1 - C_1^{dis}\Big\} + Y_2(1) \geq 0 \tag{8}$$

$$(1+r)\Bigg\{(1+r)B_1 + Y_1 - \frac{1+r}{2+r}\bigg[(1+r)B_1 + Y_1 + \frac{\mathbb{E}_1(Y_2(s))}{1+r}\bigg]\Bigg\} + Y_2(1) \geq 0 \tag{9}$$

$$(1+r)B_1 + Y_1 - \frac{1+r}{2+r}\bigg[(1+r)B_1 + Y_1 + \frac{\mathbb{E}_1(Y_2(s))}{1+r}\bigg] \geq -\frac{Y_2(1)}{1+r} \tag{10}$$

$$(2+r)B_1 + \frac{2+r}{1+r}Y_1 - (1+r)B_1 - Y_1 - \frac{\mathbb{E}_1(Y_2(s))}{1+r} \geq -\frac{2+r}{1+r}\frac{Y_2(1)}{1+r} \tag{11}$$

$$B_1 + \frac{1}{1+r}Y_1 - \frac{\mathbb{E}_1(Y_2(s))}{1+r} \geq -\frac{2+r}{1+r}\frac{Y_2(1)}{1+r} \tag{12}$$

$$(1+r)B_1 + Y_1 + \frac{2+r}{1+r}Y_2(1) \geq \mathbb{E}_1(Y_2(s)) \tag{13}$$

If this inequality does not hold, then the nonnegativity constraint on C_2 binds in at least one state of nature on date 2, so we cannot ignore the associated Kuhn-Tucker multiplier.[3] *In that case, the Kuhn-Tucker theorem predicts that date 1 consumption must make $C_2(1) = 0$ (in state 1 of date 2 when output is minimal). From using $C_2(1) = 0$ in the IBC*

$$C_2(1) = (1+r)\Big\{(1+r)B_1 + Y_1 - C_1\Big\} + Y_2(1) = 0 \tag{14}$$

we see that

$$C_1 = (1+r)B_1 + Y_1 + \frac{Y_2(1)}{1+r} \tag{15}$$

Interpretation: If the nonnegativity constraint never binds, the result from b. is still valid (logically, the bond Euler equation derived in b. must also hold). The reason is that the marginal cost associ-

[3] Assume that endowments are weakly positive in all states. Moreover, recall the simplifying assumption $Y_1 = \mathbb{E}_1(Y_2(s))$. It follows that the inequality can only be violated if $B_1 < 0$.

ated to the nonnegativity constraint is zero because the constraint is not relevant for the agents. Conversely, if the nonnegativity constraint may (!) bind, the marginal cost associated to the constraint is positive. The Euler equation derived in b. would not hold.

(d) Use the results from b. and c. to proof that consumption in period 1 is indeed smaller if the nonnegativity constraint on $C_2(s)$ may bind. Assume that $Y_1 = \mathbb{E}_1(Y_2(s))$.

Solution: *Formally, we want to proof that*

$$C_1^{dis} > C_1 \tag{16}$$

if $(1+r)B_1 + \frac{2+r}{1+r}Y_2(1) < 0$, *i.e. if the nonnegativity constraint may bind. We proof this by contracdiction. Suppose that the nonnegativity constraint never binds and* $C_1^{dis} > C_1$. *Using the results from b. and c. yields*

$$\frac{1+r}{2+r}\mathbb{E}_1\Big\{(1+r)B_1 + Y_1 + \frac{Y_2(j)}{1+r}\Big\} > (1+r)B_1 + Y_1 + \frac{Y_2(1)}{1+r} \tag{17}$$

$$(1+r)^2B_1 + (1+r)Y_1 + \mathbb{E}_1(Y_2(s)) > (2+r)(1+r)B_1 + (2+r)Y_1 + \frac{2+r}{1+r}Y_2(1) \tag{18}$$

$$(1+r^2+2r-2-2r-r-r^2)B_1 > \frac{2+r}{1+r}Y_2(1) \tag{19}$$

$$0 > (1+r)B_1 + \frac{2+r}{1+r}Y_2(1) \tag{20}$$

a contradiction (the nonnegativity constraint binds).

(e) Now assume the consumer faces complete global asset markets. $p(s) = \pi(s)$ is the price of a state contigent security which pays (1+r) if state s materializes. Find the optimal values of C_1 and $C_2(s)$ now. Why can nonnegativity constraints be disregarded in the complete markets case?

References

OBSTFELD, M., K. S. ROGOFF, ET AL. (1996): *Foundations of international macroeconomics*, vol. 30. MIT press Cambridge, MA.